A Borsuk-Ulak Theorem for Heisenberg Group Actions

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Abstract

Let $G = H_{2n+1}$ be a $(2n + 1)$-dimensional Heisenberg Lie group acts on $M = C^m - \{0\}$ and $M' = C^{m'} - \{0\}$ exponentially. By using Cohomological Index we proved the following theorem.

If $f : M \rightarrow M'$ is a $G$-equivariant map, then $m \leq m'$.

Key Words: Borsuk-Ulam Type Theorem, Cohomological Index, Group Action.

1. Introduction

By using cohomological index and relative index theories Fadell, Husseini and Rabinowitz proved Borsuk-Ulam type theorems for compact Lie groups. The ideal-valued index $\text{Index}_G(X)$ of a $G$-space $X$ for a compact Lie group, is the kernel of the map $H^*_G(\text{pt}) \rightarrow H^*_G(X)$, where $H^*_G(\text{pt})$ is the Borel cohomology of a point, which is isomorphic to $H^*(BG)$, the cohomology of the classifying space of $G$, [2,5]. If $G$ is a non-compact Lie group, where $BG$ may be acyclic, then the preceding method fails.

Fadell and Husseini introduced infinitesimal ideal-valued index theory to overcome difficulties of this type. Infinitesimal index is the kernel of the map from $BG$, the basic subcomplex of $G$ to $H^*_G(X)$, the infinitesimal $G$-deRham cohomology of a $G$-space $X$. They proved a Borsuk-Ulam type theorem for the non-compact abelian Lie group $G = C$ [3,4].

In this work we would like to extend their results to the Heisenberg groups. 

1991 AMS subject classification. 22, 55.
main theorem of this work is a Borsuk-Ulam type theorem about a Heisenberg Lie group action.

2. Preliminaries

First recall the definitions of the Lie derivative \( \Theta(X) \), the substitution operation \( i(X) \), and the differential operator \( \delta \).

The Lie derivative of a \( p \)-form \( \alpha \) with respect to \( X \in \mathfrak{X}(M) \) is the linear map \( \Theta(X) \) homogeneous of degree zero, given by

\[
\Theta(X)\alpha = X(\alpha(X_1, X_2, \ldots X_p)) - \sum_{j=1}^{p} \alpha(X_1, \ldots, [X, X_j], \ldots, X_p).
\]

The substitution operator \( i(X) \), induced by \( X \) define a \( (p-1) \)-form \( i(X)\alpha \) by

\[
i(X)\alpha(X_1, X_2, \ldots, X_{p-1}) = \alpha(X, X_1, X_2, \ldots, X_{p-1})
\]

The map \( i(X) : \Omega(M) \to \Omega(M) \) is a homogeneous operator of degree \(-1\).

The exterior derivative is the real linear map \( \delta \), homogeneous of degree 1, defined by

\[
\delta \alpha(X_0, X_1, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j X_j(\alpha(X_0, \ldots, \hat{X}_j, \ldots, X_p))
\]

\[
+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \alpha([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)
\]

where \( \alpha \) is a \( p \)-form.

Let \( G \) be a connected Lie group with its Lie algebra \( \mathfrak{g} \) and dual \( \mathfrak{g}^* \). The Weil algebra \( W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \) where \( \Lambda(\mathfrak{g}^*) \) is the exterior algebra of the dual \( \mathfrak{g}^* \) and \( S(\mathfrak{g}^*) \) is the symmetric algebra generated by elements of degree 2. Let \( s_k \) be a basis of \( S(\mathfrak{g}^*) \), \( h : \mathfrak{g}^* \to S^2(\mathfrak{g}^*) \) defined by \( h(\alpha_k) = s_k \), where \( \alpha_k \) is a basis for \( \mathfrak{g}^* \). Also we define \( \Theta_S(X)s_k = h(\Theta_E(X)\alpha_k) \), on \( S(\mathfrak{g}^*) \), where \( \Theta_E(X) \) is the usual Lie derivative defined on \( \mathfrak{g}^* \). The substitution operation \( i(X) \) is as defined above on \( \Lambda(\mathfrak{g}^*) \) and 0 on \( S(\mathfrak{g}^*) \).

The differential operator \( \delta \) on \( W(\mathfrak{g}) \) defined as follows:

\[
\delta = \delta_E + \delta_S + h
\]
\[ h = \sum_k i(X_k)\mu(h(\alpha_k)) \]

\[ \delta_E = (1/2)\sum_k \mu(\alpha_k)\Theta_E(X_k) \]

\[ \delta_S = \sum_k \mu(\alpha_k)\Theta_S(X_k) \]

\( X_k \in \mathcal{G}, \alpha_k \in \mathcal{G}^* \) and \( \mu \) is a multiplication operator defined as \( \mu(\alpha)\beta = \alpha \wedge \beta, \alpha, \beta \in \Lambda(\mathcal{G}^*) \).

Let \( L \) be a finite dimensional Lie algebra and \( i, \delta \) and \( \Theta \) are defined as above. Let \( R = \sum_{p \geq 0} R^p \) be a graded commutative algebra with differential \( \delta \).

The horizontal subalgebra of \( R \):
\[ R_{i=0} = \bigcap_{X \in L} \ker i(X). \]

The invariant subalgebra of \( R \):
\[ R_{\Theta=0} = \bigcap_{X \in L} \ker \Theta(X). \]

The basic subalgebra of \( R \):
\[ R_{i=0,\Theta=0} = (R_{i=0}) \bigcap (R_{\Theta=0}). \]

The basic subalgebra of the Weil algebra of a Lie group \( G \):
\[ W(\mathcal{G})_{i=0,\Theta=0} = (\Lambda(\mathcal{G}^*) \otimes S(\mathcal{G}^*))_{i=0,\Theta=0} = S(\mathcal{G}^*)(\Theta=0) = B\mathcal{G}. \]

The Basic Weil subalgebra serves as the algebraic analogue of the classifying space \( BG \) and we will denoted by \( B\mathcal{G} \), [3].

**Infinitesimal Index:**

The infinitesimal deRham complex of a differentiable \( G \)-manifold \( M \) is \( \Omega(M) \otimes W(\mathcal{G}) \), with \( \Theta, i \) and \( \delta \) the differential operator, where \( \Omega(M) \) denotes the differential forms on \( M \).

The basic subcomplex \( \Omega_G(M) \) of \( \Omega(M) \otimes W(\mathcal{G}) \) is defined as \( \Omega_G(M) = (\Omega(M) \otimes W(\mathcal{G}))_{i=0,\Theta=0} \) and the cohomology of \( \Omega_G(M) \) is called the infinitesimal deRham cohomology of \( M \) and
denoted by $H^*_G(M)$. The inclusion map

$$j_M : W(G) \to \Omega(M) \otimes W(G)$$

$$x \mapsto 1 \otimes x$$

$j_M$ induces morphisms

$$\tilde{j}_M : W(G)_{i=0, \theta=0} \to (\Omega(M) \otimes W(G))_{i=0, \theta=0}$$

$$\tilde{j}_M : B^\epsilon G \to \Omega_G(M)$$

a morphism $\tilde{j}_M$ of differential graded algebras is called the classifying map for the $G$-space $M$. The classifying map $\tilde{j}_M$ induces

$$\tilde{j}_M^* : B^\epsilon G \to H^*_G(M)$$

since $\delta = 0$ on $S(G^*)$.

The infinitesimal $G$-index of $M$, $\text{Index}_G M$, is the kernel of the map

$$j_M^* : B^\epsilon G \to H^*_G(M)$$

where $j_M^*$ is induced by $\tilde{j}_M : B^\epsilon G \to \Omega_G(M)$.

The infinitesimal $G$-index possesses the following properties:

**Continuity.** [3] If $B^\epsilon G$ is Noetherian, there is an open $G$-set $V_0$ such that $X \subset V_0$ and for every open $G$-set $U$, $X \subset U \subset V_0$,

$$\text{Index}_G X = \text{Index}_G U.$$ 

**Monotonicity.** [3] Let $B^\epsilon G$ be Noetherian, and $f : M \to N$ is a differentiable $G$-map, $X \subset M$ and $Y \subset N$ are $G$-subsets with $f(X) \subset Y$, then

$$\text{Index}_G Y \subset \text{Index}_G X.$$
Additivity. [3] Let \( X \cup Y \subset M \), \( X \) and \( Y \) be \( G \)-sets and \( BG \) is Noetherian, then

\[
(\text{Index}_G X)(\text{Index}_G Y) \subset \text{Index}_G (X \cup Y).
\]

Recall that, by definition, the \((2n + 1)\)-dimensional Heisenberg Lie group \( H_{2n+1} \) is the Lie group of real matrices of the form:

\[
\begin{pmatrix}
1 & x_1 & x_2 & \cdots & x_n & z \\
0 & 1 & 0 & \cdots & 0 & y_1 \\
0 & 0 & 1 & \cdots & 0 & y_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & y_n \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

where \( x_i, y_i \), and \( z \in R \).

\( H_{2n+1} \) is a two-step, nilpotent Lie group. Let \( \mathcal{H}_{2n+1} \) denote the Lie algebra of \( H_{2n+1} \). \( \mathcal{H}_{2n+1} \) is generated by \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \) with all its commutators equal to zero except \( [X_i, Y_i] = Z, i = 1, 2, \ldots, n \). The dual \( \mathcal{H}_{2n+1}^* \) is generated by \( \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma\} \), where

\[
X_i = \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}
\]

and \( \alpha_i = dx_i, \beta_i = dy_i, \gamma = dz - \sum_{i=1}^n x_i dy_i, i = 1, \ldots, n \).

**Proposition 1.** Let \( G \) be a \((2n + 1)\)-dimensional Heisenberg Lie group and \( A = G/[G, G] \) be its abelinization, then \( BG \cong BA \). Where \( BA \) is the polynomial algebra in \( s_1, \ldots, s_n, t_1, \ldots, t_n \), and \( s_i = h(\alpha_i), t_i = h(\beta_i) \).

**Proof.** The basic subcomplex \( BG \cong (SG^*)_{\Theta = 0} \), where \( \Theta = 0 \) means that the Lie derivatives is zero with respect to all \( X \in \mathcal{G} \). Let \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \), and \( \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma\} \) denote the generators of \( \mathcal{G} \) and \( \mathcal{G}^* \) respectively, then \( A \) and \( A^* \) are generated by \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \), and \( \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \) respectively. It is known that, if \( A \) is abelian, all the generators of \( A^* \) are invariant differential forms, then \( \Theta(X) \alpha_i = \Theta(X) \beta_j = 0 \) for all \( X \in \mathcal{G} \) and \( 1 \leq i, j \leq n \), [3]. Therefore

\[
BA = (SA^*)_{\Theta = 0} = SA^* \subset (SG^*)_{\Theta = 0} = BG
\]
Now assume that $BG$ has some elements $\omega$ which is not an element of $BA$. Then $\omega$ contains a polynomial of $r = h(\gamma)$. Let

$$\varphi(r) = \sum_{i=0}^{m} a_ir^i \neq 0$$

where $a_i$’s are linearly independent in $BG$.

$$\omega = \sum_{j=1}^{k} c_js_j \wedge \ldots \wedge s_j \wedge t_{j_1} \wedge \ldots t_{j_p} \wedge \varphi(r)$$

and $\Theta(X)\omega = 0$ for all $X \in G$. Since $\Theta(X)s_i = h(\Theta(X)a_i) = 0$ and $\Theta(X)t_j = h(\Theta(X)\beta_j) = 0$,

$$\Theta(X)\omega = \left(\sum_{j=1}^{k} c_js_j \wedge \ldots \wedge s_j \wedge t_{j_1} \wedge \ldots t_{j_p} \wedge \Theta(X)\varphi(r)\right) = 0$$

$$\Theta(X)\varphi(r) = 0$$

$$a_1(\Theta(X)r) + a_2(\Theta(X)r^2) + \cdots + a_m(\Theta(X)r^m) = 0$$

$$a_1(\Theta(X)r) + 2a_2r(\Theta(X)r) + \cdots + ma_mr^{m-1}(\Theta(X)r) = 0$$

$$a_1 + 2a_2r + \cdots + ma_mr^{m-1}(\Theta(X)r) = 0$$

since $\Theta(X)r = h(\Theta(X)\gamma) \neq 0$, then

$$a_1 + 2a_2r + \cdots + ma_mr^{m-1} = 0$$

by linearly independence, $a_i = 0$ for $i = 1, \ldots, m$, thus $\varphi(r) = a_0$. Therefore, $BG \cong BA$. □

**Exponential $G$ Action:**

Let $C^n$ denote the complex $n$-space and $M = C^n$. Fadell and Husseini defined the right $C$-action

$$M \times G \rightarrow M$$

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irrational for $i$ of non-zero real numbers such that $\lambda_i/\lambda_j$ are irrational for $i \neq j$. This action takes $C^n = \{0\}$ onto itself. Fadell and Husseini called this action an exponential action with parameters $\lambda_1, \ldots, \lambda_n$, and then proved the following Borsuk-Ulam type theorem,[3]:

If $G = C$ acts on $C^n$ and $C^m$ with exponential actions with parameters $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, respectively, with $m < n$. Then every $G$-map $f : C^n \to C^m$ has a non-trivial zero. Alternatively, there does not exist a $G$-map $f : C^n \to C^m - \{0\}$.

Here we want to use the same action for $G = G_1 \oplus \cdots \oplus G_n$ and $M = C^m - \{0\} = (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\}$ where $G_k = R + iR$ and $M_k = C^{m_k} - \{0\}$.

The $G_k$ action on $M_k$ is defined as follows:

$$
\varphi_k : (\tilde{z}, \xi, \tilde{\lambda}_k) \mapsto e^{\xi_k} (z_1 (e^{iy_k})^{\lambda_1}, \ldots, z_n (e^{iy_k})^{\lambda_n}).
$$

where $\xi_k = x_k + iy_k \in G_k$, $\tilde{z} = (z_1, \ldots, z_m) \in M_k$, and $\tilde{\lambda}_k = (\lambda_1, \ldots, \lambda_m)$ is an $m_k$-tuple of non-zero real numbers such that $\lambda_i/\lambda_j$ are irrational for $i \neq j$.

Since $z_j (e^{iy_k})^{\lambda_i} = z_j$ for all $j$, if and only if $\lambda_j y_k = 2q\pi$, $q \in \mathbb{Z}$, the discrete subgroup $\Gamma_j = \{(2q\pi/\lambda_j)\}$, $q \in \mathbb{Z}$ appears as non-trivial isotropy. If we require that $\lambda_i/\lambda_j$ be irrational for $i \neq j$, then these would be the only non-trivial isotropy subgroups. We also note that the representation $iy_k \to diag((e^{iy_k})^{\lambda_1}, \ldots, (e^{iy_k})^{\lambda_m})$ has compact image $R/D = S^1$, $D$ discrete, in $U(m_k)$ if all ratios $\lambda_i/\lambda_j$ are rational. Otherwise, this imbedding of $R$ in $U(m_k)$ has closure which is a torus of dimension $\geq 2$.

The $G$-action on $M$ is defined as follows:

$$
M \times G \to M
$$

$$
(\oplus_{k=1}^n M_k) \times (\oplus_{k=1}^n G_k) \to (\oplus_{k=1}^n M_k)
$$

$$
((z_1, \ldots, z_n), (\xi_1, \ldots, \xi_n), (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)) \mapsto e^{(\sum_{j=1}^n \xi_j)} (\varphi_1(z_1, iy_1, \tilde{\lambda}_1), \ldots, \varphi_n(z_n, iy_n, \tilde{\lambda}_n)).
$$

Here we have

$$
G = (G_1 \oplus \cdots \oplus G_n) \oplus iG_1 \oplus \cdots \oplus iG_n.
$$

This can be written as,

$$
G = (G_1 \oplus \cdots \oplus G_n) \oplus iG_1 \oplus \cdots \oplus iG_n
$$

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Let $G' = R$ and $G'' = R_1 \oplus \cdots \oplus R_n$. Here $G'$ acts freely on $C^m - \{0\}$ and $C^m - \{0\} \cong R^n \times S^{2m-1}$ with orbit space $(C^m - \{0\})/G'' \cong S^{2m-1}$. If $\Gamma$ is a discrete subgroup of $G'$, note that $G'/\Gamma = S^1$ is a compact group and

$$M/\Gamma = (C^m - \{0\})/\Gamma \cong S^1 \times S^{2m-1}.$$ 

By lemmas 4.2, 4.3, and 4.4 of Fadell and Husseini [3], there is a natural chain equivalence

$$\nu : \Omega(M)_{\Theta} \cong \Omega(S^1 \times S^{2m-1}).$$

Now, consider $M = C^m - \{0\}$ as a $G'$-space by restricting the $G$ action. Then the natural projection $M \rightarrow M/G' = S^{2m-1}$ is a locally trivial principle $G$-bundle by Palais' theorems [7].

**Proposition 2.** [3] There is a chain equivalence $\gamma : \Omega_{G'}(M) \rightarrow \Omega(S^{2m-1})$.

Atiyah and Bott showed that, since torus $T$ is compact, the Borel cohomology $H^*_T(S^{2m-1}; R)$ is naturally isomorphic to the infinitesimal cohomology $H^*_T(S^{2m-1})$ [1]. Furthermore, the ideal-valued index, $\text{Index}_T(S^{2m-1})$ and the infinitesimal ideal-valued index, $\text{Index}_T(S^{2m-1})$ coincide when $H^*(BT; R)$ and $BT$ are naturally identified. The inclusion map $T \subset T^m \subset U(m)$ induces homomorphisms $\lambda_j : T \rightarrow S^1$, $j = 1, 2, \ldots, m$, and if $S$ is the Lie algebra of $S^1$, $\lambda_j$ induces $\lambda'_j : BS \rightarrow BT$. If $\sigma$ is the generator of $BS$ set $\lambda'_j = \lambda'_j(\sigma)$. Then, the natural inclusion

$$BT \rightarrow \Omega(M)_{\Theta} = BT$$

induces a surjection

$$BT \rightarrow H^*_T(S^{2m-1})$$

with kernel $P_T$ the principal ideal generated by $\varepsilon = \lambda'_1 \lambda'_2 \cdots \lambda'_m$. Here each $\lambda_j : T \rightarrow S^1$ is nontrivial because $(S^{2m-1})^{G''} = (S^{2m-1})^T = 0$. This implies that if $g : BT \rightarrow BG''$ is induced by inclusion $g(\lambda'_j) \neq 0$ for $j = 1, 2, \ldots, m$. 

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Lemma. \([a] \quad g(\varepsilon), \) is a polynomial of \(t_1, \ldots, t_n,\) of degree \(m.\)

3. Results

Now consider the exponential \(G = G_1 \oplus \cdots \oplus G_n\) action on

\[ M = (C^{m_1} \oplus \cdots \oplus C^{m_n}) \setminus \{0\} = C^{m} \setminus \{0\} \]

where \(m = m_1 + \cdots + m_n,\) and \(M = C^{m} \setminus \{0\} = R^+ \times S^{2m-1}.\)

**Theorem 1.** Let \(G = G_1 \oplus \cdots \oplus G_n\) acts on \(M = (C^{m_1} \oplus \cdots \oplus C^{m_n}) \setminus \{0\}\) with an exponential action with parameters \((\lambda_1, \ldots, \lambda_m).\) Then, the following inclusion map

\[ j_M : W(G) \rightarrow \Omega_G(M) \]

induces a surjection

\[ j_M^* : B_G \rightarrow H_G^*(M) \]

The kernel of this map is an ideal generated by \(s_1 + \cdots + s_n\) and \(\lambda_1 \lambda_2 \ldots \lambda_m.\)

**Proof.** I. First compare the spectral sequences \(\Omega(M)_{\Theta = 0} \otimes S(G^*)\) and \(\Omega(M)_{\Theta' = 0} \otimes S(G'^*)\)

via the filtration preserving map

\[ \Omega(M)_{\Theta = 0} \otimes S(G^*) \rightarrow \Omega(M)_{\Theta' = 0} \otimes S(G'^*) \]

induced by \(G' \subset G.\) Induced map on fibers \(\Omega(M)_{\Theta = 0} \rightarrow \Omega(M)_{\Theta' = 0}\) is a chain equivalence and on the base \(S(G^*) = R[s_1, \ldots, s_n, t_1, \ldots, t_n] \rightarrow R[s_1 + \cdots + s_n] = S(G'^*).\) At \(E_2\)-level we have the following diagram:

\[
\begin{array}{ccc}
H^*(\Omega(M)_{\Theta = 0}) \otimes S(G^*) & \rightarrow & H^*(\Omega(M)_{\Theta' = 0}) \otimes S(G'^*) \\
\downarrow d_2 & & \downarrow d'_2 \\
H^*(\Omega(M)_{\Theta = 0}) \otimes S(G^*) & \rightarrow & H^*(\Omega(M)_{\Theta' = 0}) \otimes S(G'^*)
\end{array}
\]

Let \(u^* \in H^1(\Omega(M)_{\Theta' = 0}) = H^1(S^1 \times S^{2m-1})\) denotes a generator corresponding to the \(S^1\)-factor. Since \(H^*_G(M) = H^*(S^{2m-1})\) then \(d'_2 u^* \neq 0.\) We may assume without loss.
that, if \( u \in H^1(\Omega(M)_{t=0}) = H^1(S^1 \times S^{2m-1}) \) is the generator corresponding to \( u' \), then \( d_2 u = s_1 + \cdots + s_n \).

II. Now compare the spectral sequences for \( \Omega(M)_{t=0} \otimes S(G^*) \) and \( \Omega(S^{2m-1})_{t=0} \otimes S(G^{**}) \) via natural map

\[
\Omega(M)_{t=0} \otimes S(G^*) \rightarrow \Omega(S^{2m-1})_{t=0} \otimes S(G^{**})
\]

induced by \( M = C^m - \{0\} \rightarrow S^{2m-1} \) and \( G'' \subset G \).

If we take a generator \( v'' \in H^*(\Omega(S^{2m-1})_{t=0}) \supseteq H^*(S^{2m-1}) \) and apply Husseini's Lemma, we will see that \( d_m v'' = C \lambda_1' \lambda_2' \cdots \lambda_m' \), \( C \neq 0 \). Since \( v'' \) may be chosen as the image of \( v \), where \( v \in H^{2m-1}(\Omega(M)_{t=0}) \), which denotes a generator corresponding to \( S^{2m-1} \) factor, then we have \( d_m v = C \lambda_1 \lambda_2 \cdots \lambda_m \), \( \square \).

Now assume \( G \cong \mathbb{R}^{2n} \), and \( M = C^m - \{0\} \subset (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\} \).

Let \( \varphi : G \rightarrow \text{Gl}_m(C) \), where \( m = m_1 + \cdots + m_n \) be a homomorphism. Also assume that \( \text{im} \varphi \subset \text{diagonal matrices} \) and \( \text{closure}(\varphi(G)) = R_1 \oplus R_1 \oplus \cdots \oplus R_n \oplus iR_n \).

**Corollary 1.** Let \( G \cong \mathbb{R}^{2n} \) and \( M = C^m - \{0\} \), with exponential \( G \) action given as above. Then \( \text{Index}_G(M) = < s_1 + \cdots + s_n, \lambda_1' \lambda_2' \cdots \lambda_m' > \).

Now consider \( G = H_{2n+1} \), \( (2n+1) \)-dimensional nilpotent Heisenberg Lie group of real matrices and let

\[ \psi : G \rightarrow G/[G,G] \cong \mathbb{R}^{2n} \]

and \( M = C^m - \{0\} \subset (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\} \), and \( G \) acts on \( M \) via \( \psi \).

\[
M \times G \rightarrow M
\]

\[
((z_1, \ldots, z_n), g) \rightarrow e^{\sum_{j=1}^{n} z_j l_j}(\varphi_1(z_1, iy_1, \tilde{l}_1), \ldots, \varphi_n(z_n, iy_n, \tilde{l}_n))
\]

**Proposition 3.** Let \( G \) be a \( (2n+1) \)-dimensional nilpotent Heisenberg Lie group and \( A = G/[G,G] \) be its abelianization and \( M = C^m - \{0\} \), the complex \( m \)-space. If the \( G \)-action \( M \times G \rightarrow M \) is defined as above, then

\[
H^*_G(M) \cong H^*_A(M).
\]

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Proof. Let \( X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z \) and \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma \) denote the generators of \( G \), and \( G^* \) respectively, then \( A \) and \( A^* \) are generated by \( \{ X_1, \ldots, X_n, Y_1, \ldots, Y_n \} \) and \( \{ \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \} \) respectively. Here \( SA^* \) is generated by \( \{ h(\alpha_1), \ldots, h(\alpha_n), h(\beta_1), \ldots, h(\beta_n) \} = \{ s_1, \ldots, s_n, t_1, \ldots, t_n \} \) also \( SG^* \) is generated by \( \{ s_1, \ldots, s_n, t_1, \ldots, t_n, r \} \) where \( r = h(\gamma) \).

We want to show that \( \Omega_G(M) \cong \Omega_A(M) \) where
\[
\Omega_G(M) = (\Omega(M) \otimes S(G^*))_{\Theta_G = 0}
\]
and
\[
\Omega_A(M) = (\Omega(M) \otimes S(A^*))_{\Theta_A = 0}.
\]

We need to check that \( \Theta_X \), for \( X \in \{ \text{Center of } G \} \). Since the center of \( G \) is generated by \( Z \), \( \Theta_Z = 0 \) on \( \Omega(M) \) and also \( S(G^*)_{\Theta_x = 0} = S(A^*) \), then
\[
(\Omega(M) \otimes S(G^*))_{\Theta_Z = 0} = (\Omega(M) \otimes S(A^*)).
\]
Since
\[
(\Omega(M) \otimes S(G^*))_{\Theta_G = 0} = (\Omega(M) \otimes S(G^*))_{\Theta_Z = 0, \Theta_A = 0}
\]
then
\[
(\Omega(M) \otimes S(G^*))_{\Theta_G = 0} = (\Omega(M) \otimes S(A^*))_{\Theta_A = 0}.
\]
This gives us
\[
\Omega_G(M) \cong \Omega_A(M)
\]
and then
\[
H^*_G(M) \cong H^*_A(M).
\]

Proposition 4. Let \( G = H_{2n+1} \) acts on \( M = (C^{m_1} \oplus \cdots \oplus C^{m_n}) \setminus \{0\} \) with an exponential action with parameters \( \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \). Then,
Index\textsubscript{G}(M) = \langle s_1 + \cdots + s_n, \lambda'_1 \lambda'_2 \cdots \lambda'_m \rangle.

\textbf{Proof.} By Proposition 1. and Proposition 3. \(BG \cong BA\) and \(H_G(M) \cong H_A(M)\). Then,

\[ \text{Index}_A(M) = \text{Index}_G(M). \]

The Proposition follows since \(\text{Index}_A(M) = \langle s_1 + \cdots + s_n, \lambda'_1 \lambda'_2 \cdots \lambda'_m \rangle\) from Proposition 3. \(\square\)

We may now give the following Borsuk-Ulam type theorem.

\textbf{Theorem 2.} Let \(G = H_{2n+1}\) acts on \(M = C^p - \{0\} = (C^{p_1} \oplus \cdots \oplus C^{p_n}) - \{0\}\) and \(N = C^q - \{0\} = (C^{q_1} \oplus \cdots \oplus C^{q_n}) - \{0\}\) with an exponential actions with parameters \(\{\lambda_1, \lambda_2, \ldots, \lambda_p\}\) and \(\{\mu_1, \mu_2, \ldots, \mu_q\}\) respectively. If \(f : M \to N\) is a \(G\)-equivariant map, then \(p \leq q\).

\textbf{Proof.} Proposition 4. gives that

\[ \text{Index}_G(M) = \langle s_1 + \cdots + s_n, \lambda'_1 \lambda'_2 \cdots \lambda'_p \rangle \]

and

\[ \text{Index}_G(N) = \langle s_1 + \cdots + s_n, \mu'_1 \mu'_2 \cdots \mu'_q \rangle. \]

Where \(\lambda'_1 \lambda'_2 \cdots \lambda'_p\) and \(\mu'_1 \mu'_2 \cdots \mu'_q\) are polynomials of \(t_1, \ldots, t_n\) with degrees \(p\) and \(q\) respectively. By monotonicity of Index\textsubscript{G}, if \(f : M \to N\) is a \(G\)-equivariant map, then

\[ \text{Index}_G(M) \supset \text{Index}_G(N). \]

This implies that the degree of \(\lambda'_1 \lambda'_2 \cdots \lambda'_p\) is smaller than or equal to the degree of \(\mu'_1 \mu'_2 \cdots \mu'_q\).

Thus \(p \leq q\). \(\square\)

\textbf{Acknowledgement}

Author would like to thank his Ph.D. adviser Prof. Suifian Y. Husseini and Prof. Edward Fadell for their help and encouragement during his studies at University of Wisconsin-Madison.

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Received 21.06.2000