Turk J Math
24 (2000) , 283 – 294.

Intrinsic Equations for a Relaxed Elastic Line on an Oriented Hypersurface in the Minkowski Space $\mathbb{R}_1^n$

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Abstract

We gived the intrinsic equations for a relaxed elastic line on an oriented surface in $\mathbb{R}_1^3$ ([1],[2]).

In this paper, we derived the intrinsic equations for a relaxed elastic line on an oriented time-like hypersurface and space-like hypersurface in the Minkowski space $\mathbb{R}_1^n$ and gived additional results about relaxed elastic lines on various timelike and spacelike hypersurface in the Minkowski space $\mathbb{R}_1^n$.

Key Words: Elastic line, Minkowski space.

1. Introduction

In this section, we give some fundamental definitions and theorems.

Definition 1.1. Let α denote an arc on a connected oriented hypersurface M in $\mathbb{R}_1^n$ parametrized by arc length s, $0 \leq s \leq l$. Let $k_1(s)$ be the curvature of the first curvature of $\alpha(s)$. The first total square curvature K of α in $\mathbb{R}_1^n$ is defined by

$$K = \int_0^l k_1^2 ds. \tag{1.1}$$

Definition 1.2. The arc α is called a relaxed elastic line if it is an extremal for the variational problem of minimizing the value of K within the family of all arcs of length l on M having the same initial point and initial direction as α in the Minkowski space $\mathbb{R}_1^n$.

Definition 1.3. On an $n \times n$ matrix, the following conditions are equivalent:

(1) g$\epsilon O_{\nu}(n)$

(2) g$^{t} = \varepsilon g^{-1} \varepsilon$

(3) The columns(rows) of g form an orthonormal basis for $\mathbb{R}^n_\nu$ (first ν vectors are timelike).

(4) g carries one (hence every) orthonormal basis for $\mathbb{R}^n_\nu$ to an orthonormal basis [3].

Definition 1.4. Let M be a pseudo-Euclidean hypersurface in $\mathbb{R}^n_1$ and a curve α which lies on M. Apart from the Frenet vector field system $\{V_1, \overline{V_2}, \overline{V_3}..., \overline{V_{n-1}}, \overline{V_n}\}$, there is also exist a second orthonormal vector field system $\{V_1, ..., V_{n-1}, N\}$ at every point of the curve α. At a point $\alpha(s)$ of α, let $V_1(s) = \alpha'(s)$ denote the unit tangent vector to α, let $N(s)$ denote the unit hypersurface normal to M. $\{V_1, ..., V_{n-1}, N\}$ gives a basis for all vectors at $\alpha(s)$ and $\{V_1, ..., V_{n-1}, N\}$ gives a basis for the vectors tangent to M at $\alpha(s)$. Let II denote the second fundamental form of M. The orthonormal system $\{V_1, ..., V_{n-1}, N\}$ is called natural frame field for hypersurface strip (α, M) .

Definition 1.5. Let M be a pseudo-Euclidean hypersurface in $\mathbb{R}^n_1$ and a curve α be a curve on M. Then, for each i, $1 \leq i \leq n-1$, the function

$$k_{ig} : \ I \subset R \quad \rightarrow R$$

defined for s$\in I$ by

$$k_{ig}(s) =< V_i'(s), V_{i+1}(s) >$$

is called the ith geodesic curvature function of the curve α and $k_{ig}(s)$ is called the ith geodesic curvature of the curve α at $\alpha(s)$ in $\mathbb{R}^n_1$.

Theorem 1.1. Let M be a pseudo-Euclidean hypersurface in $\mathbb{R}^n_1$ and α denote an arc on M. The derivative formulas of orthonormal vector field system $\{V_1, ..., V_{n-1}, N\}$ is

$$\begin{bmatrix} V_1' \\ V_2' \\ . \\ . \\ . \\ V_{n-1}' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_{1g} & 0 & ... & 0 & \varepsilon_n a_1 \\ -\varepsilon_1 k_{1g} & 0 & \varepsilon_3 k_{2g} & ... & 0 & \varepsilon_n a_2 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & \varepsilon_n a_{n-1} \\ -\varepsilon_1 a_1 & -\varepsilon_2 a_2 & -\varepsilon_3 a_3 & ... & -\varepsilon_{(n-1)} a_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ . \\ V_{n-1} \\ N \end{bmatrix}, \tag{1.2}$$

where k_{ig} is the ith geodesic curvature funtion,

$$a_i = II(V_1, V_i), 1 \leq i \leq n-1$$

and

$$< V_1, V_1 >= \varepsilon_1, < V_2, V_2 >= \varepsilon_2 \ ,..., < N, N >= \varepsilon_n.$$

2. Obtaining the Equations

Now, assume that α lies in a coordinate patch $(u_1, ..., u_{n-1}) \to x(u_1, ..., u_{n-1})$ of M and let $x_{u_1} = \dfrac{\partial x}{\partial u_1}, x_{u_2} = \dfrac{\partial x}{\partial u_2}, ..., x_{u_{n-1}} = \dfrac{\partial x}{\partial u_{n-1}}$. Then α is expressed as

$$\alpha(s) = x\left(u_1(s), u_2(s), u_3(s), ..., u_{n-1}(s)\right), \quad 0 \leq s \leq l$$

with

$$V_1(s) = \alpha'(s) = x_{u_1}\frac{du_1}{ds} + x_{u_2}\frac{du_2}{ds} + ... + x_{u_n}\frac{du_n}{ds}$$

and

$$V_2(s) = p_1(s)x_{u_1} + p_2(s)x_{u_2} + ... + p_{n-1}(s)x_{u_{n-1}}$$

for suitable scalar functions $p_1(s)$, $p_2(s)$,...,$p_{n-1}(s)$.

Next, we must define variational fields for our problem. In order to obtain variational arcs of length l, it is generally necessary to extend α to an arc α^* defined for $0 \leq s \leq l^*$, with $l^* > l$, but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \leq s \leq l^*$, be a scalar function of class C^{n-1}, not vanishing identically. Define

$$\eta_1(s) = \mu(s)p_1^*(s) \ , \ \eta_2(s) = \mu(s)p_2^*(s), ..., \eta_{n-1}(s) = \mu(s)p_{n-1}^*(s).$$

Then, along α

$$\eta_1(s)x_{u_1} + \eta_2(s)x_{u_2} + ... + \eta_{n-1}(s)x_{u_{n-1}} = \mu(s)V_2(s). \tag{2.1}$$

Assume also that

$$\mu(0) = 0, \mu'(0) = 0. \tag{2.2}$$

Now define

$$\beta(\sigma; t) = x\left(u_1(\sigma) + t\eta_1(\sigma), ..., u_{n-1}(\sigma) + t\eta_{n-1}(\sigma)\right), \tag{2.3}$$

for $0 \leq \sigma \leq l^*$. For $|t| < \varepsilon$ (where $\varepsilon > 0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma; t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma; t)$ gives an arc with the same initial point and initial direction as α, because of (2.2). For $t = 0$, $\beta(\sigma; 0)$ is the same as α^* and σ is arc length. For $t \neq 0$, the parameter σ is not arc length in general.

For fixed t, $|t| < \varepsilon$, let $L^*(t)$ denote the length of the arc $\beta(\sigma; t), 0 \leq \sigma \leq l^*$. Then

$$L^*(t) = \int_0^{l^*} \sqrt{\left|\left\langle \frac{\partial \beta}{\partial \sigma}(\sigma; t), \frac{\partial \beta}{\partial \sigma}(\sigma; t) \right\rangle\right|} d\sigma \tag{2.4}$$

with

$$L^*(0) = l^* > l. \tag{2.5}$$

It is clear from (2.3) and (2.4) that $L^*(t)$ is continuous. In particular, it follows from (2.5) that

$$L^*(t) > \frac{l + l^*}{2} > l, \qquad (|t| < \varepsilon_* \ ) \tag{2.6}$$

for a suitable ε_* satisfying $0 < \varepsilon_* \leq \varepsilon.$ Because of (2.6), we can restrict $\beta(\sigma; t)$, $0 \leq |t| < \varepsilon_*,$ to an arc of length l by restricting the parameter σ to an interval $0 \leq \sigma \leq \lambda(t) \leq l^*$, by requiring

$$\int_0^{\lambda(t)} \sqrt{\left|\left\langle \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right\rangle\right|} d\sigma = l. \tag{2.7}$$

Note that $\lambda(0) = l$. The function $\lambda(t)$ need not be determined explicitly, but we shall need

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \varepsilon_1 \int_0^l \mu k_{1g} ds. \tag{2.8}$$

The proof of (2.8) and of other results below will depend on calculations from (2.3) such as

$$\left.\frac{\partial \beta}{\partial \sigma}\right|_{t=0} = V_1, \qquad 0 \leq \sigma \leq l \tag{2.9}$$

which gives

$$\left.\frac{\partial^2 \beta}{\partial \sigma^2}\right|_{t=0} = V_1' = \varepsilon_2 k_{1g} V_2 + \varepsilon_n a_1 N. \tag{2.10}$$

Also, it follows from (2.1) that

$$\left.\frac{\partial \beta}{\partial t}\right|_{t=0} = \mu V_2. \tag{2.11}$$

Using (2.1), the second differentiation of (2.11) gives

$$\left.\frac{\partial^2 \beta}{\partial t \partial \sigma}\right|_{t=0} = -\varepsilon_1 \mu k_{1g} V_1 + \mu' V_2 + \varepsilon_3 \mu k_{2g} V_3 + \varepsilon_n \mu a_2 N \tag{2.12}$$

and the third differentiation of (2.11) gives

$$\left.\frac{\partial^3\beta}{\partial t\partial\sigma^2}\right|_{t=0} \begin{aligned}[t] &= \left(-2\varepsilon_1\mu' k_{1g} - \varepsilon_1\mu k'_{1g} - \varepsilon_1\varepsilon_n\mu a_1 a_2\right) V_1 \\ &+ \left(\mu'' - \varepsilon_1\varepsilon_2\mu k_{1g}^2 - \varepsilon_2\varepsilon_3\mu k_{2g}^2 - \varepsilon_2\varepsilon_n\mu a_2^2\right) V_2 \\ &+ \left(2\varepsilon_3\mu' k_{2g} + \varepsilon_3\mu k'_{2g} - \varepsilon_3\varepsilon_n\mu a_2 a_3\right) V_3 \\ &+ (\varepsilon_3\varepsilon_4\mu k_{2g}k_{3g} - \varepsilon_4\varepsilon_n\mu a_2 a_4) V_4 \\ &- (\varepsilon_5\varepsilon_n\mu a_2 a_5 V_5 + \varepsilon_6\varepsilon_n\mu a_2 a_6 V_6 + ... + \varepsilon_{n-1}\varepsilon_n\mu a_2 a_{n-1} V_{n-1}) \\ &+ (-\varepsilon_1\varepsilon_n\mu k_{1g} a_{1+} 2\varepsilon_n\mu' a_2 + \varepsilon_3\varepsilon_n\mu k_{2g} a_3 + \varepsilon_n\mu a'_2) N. \end{aligned} \tag{2.13}$$

To prove (2.8), differentiate (2.7) with respect to t, remembering that l is constant, and evaluate at t=0 using (2.9) and (2.12), with $\lambda(0) = l$.

$$\left.\frac{d\lambda}{dt}\right|_{t=0}\sqrt{\left|\left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}\right\rangle\right|} + \int_0^l \left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0}\right\rangle \frac{\sqrt{\left|\left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}\right\rangle\right|}}{\left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}\right\rangle} ds = 0$$

Now, let $K(t)$ denote the total square curvature of the arc $\beta(\sigma;t)$, $0 \leq \sigma \leq \lambda(t)$, $|t| < \varepsilon_*$. Since σ is not generally arc length for $t \neq 0$, the total square curvature is ,

$$K(t) = \int_0^{\lambda(t)} \frac{\left|\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma,t)\wedge\frac{\partial^2\beta}{\partial\sigma^2}(\sigma,t), \frac{\partial\beta}{\partial\sigma}(\sigma,t)\wedge\frac{\partial^2\beta}{\partial\sigma^2}(\sigma,t)\right\rangle\right|}{\left|\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma,t), \frac{\partial\beta}{\partial\sigma}(\sigma,t)\right\rangle\right|^3} \left|\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma,t), \frac{\partial\beta}{\partial\sigma}(\sigma,t)\right\rangle\right|^{1/2} d\sigma.$$

A necessary condition for α being extremal is that $K'(0) = 0$ for arbitrary μ satisfying (2.2). In calculating $K'(t)$, we give explicitly only those terms which do not vanish for $t = 0$. The omitted terms are those with a factor $\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma}\right\rangle$, which vanishes at $t = 0$, since $< V_1', V_1 >= 0$. Thus

$$K'(t) = \frac{d\lambda}{dt}\left\{\left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-3/2} \left| - \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|\right\}_{\sigma=\lambda(t)}$$

$$-3\int_0^{\lambda(t)} \left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-5/2} \left\langle \frac{\partial^2\beta}{\partial t\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle \frac{\left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle} \left|\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right| d\sigma$$

$$+2\int_0^{\lambda(t)} \left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-3/2} \left\langle \frac{\partial^3\beta}{\partial t\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle \frac{\left|\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|}{\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle} d\sigma + ...$$

Using (2.8), (2.9), (2.12) and (2.10), we find

$$\begin{aligned} K'(0) = \; & \varepsilon_1 \int_0^l \mu k_{1g} ds \left\{\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right|\right\}_{\sigma=\lambda(0)} \\ & +2\int_0^l k_{1g}\left(\mu'' - \varepsilon_1\varepsilon_2\mu k_{1g}^2 - \varepsilon_2\varepsilon_3\mu k_{2g}^2 - \varepsilon_2\varepsilon_n\mu a_2^2\right) \frac{\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right|}{\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2} ds \\ & +2\int_0^l a_1\left(-\varepsilon_1\varepsilon_n\mu k_{1g} a_1 + 2\varepsilon_n\mu' a_2 + \varepsilon_3\varepsilon_n\mu k_{2g} a_3 + \varepsilon_n\mu a_2'\right) \frac{\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right|}{\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2} ds \\ & +3\varepsilon_1 \int_0^l \mu k_{1g}\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right| ds. \end{aligned} \tag{2.14}$$

However, with integration by parts and (2.2),

$$2\int_0^l \mu'' k_{1g} ds = 2\mu'(l)k_{1g}(l) - 2\mu(l)k_{1g}'(l) + 2\int_0^l \mu k_{1g}'' ds \tag{2.15}$$

and

$$4\int_0^l \mu' a_1 a_2 ds = 4\mu(l)a_1(l)a_2(l) - 4\int_0^l \mu a_1' a_2 ds - 4\int_0^l \mu a_1 a_2' ds. \tag{2.16}$$

2.1. Intrinsic equations for a relaxed elastic line on a timelike hypersurface

If V_1 is timelike, $V_2, V_3, ..., V_{n-1}$ and N are spacelike then

$< V_1, V_1 >= \varepsilon_1 = -1, \quad < V_2, V_2 >= \varepsilon_2 = 1, ..., < N, N >= \varepsilon_n = 1.$

In the case of $k_{1g}^2 > a_1^2$,

$$\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right| = k_{1g}^2 + a_1^2. \tag{2.17}$$

Substituting (2.8), (2.9), (2.12), (2.13), (2.15), (2.16) and (2.17) in (2.14), we find

$$\begin{aligned} K'(0) = \; & \int_0^l \mu\{2k''_{1g} - 2a_1a'_2 - 4a_2a'_1 + 2k_{2g}a_1a_3 \\ & + k_{1g}\left(-k_{1g}^2(l) - a_1^2(l) - k_{1g}^2 - a_1^2 - 2k_{2g}^2 - 2a_2^2\right)\}ds \\ & + 2\mu'(l)k_{1g}(l) - 2\mu(l)k'_{1g}(l) + 4\mu(l)a_1(l)a_2(l). \end{aligned}$$

In order that $K'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given timelike arc α must satisfy two boundary conditions and differential equation:

$$\begin{aligned} &(1) \quad k_{1g}(l) = 0 \\ &(2) \quad k'_{1g}(l) = 2a_1(l)a_2(l) \\ &(3) \quad 2k''_{1g} - 2a_1a'_2 - 4a_2a'_1 + 2k_{2g}a_1a_3 \\ &\qquad + k_{1g}\left(-a_1^2(l) - k_{1g}^2 - a_1^2 - 2k_{2g}^2 - 2a_2^2\right) = 0. \end{aligned} \tag{2.18}$$

2.2. Intrinsic equations for a relaxed elastic line on an spacelike hypersurface

If V_1, $V_2, ..., V_{n-1}$ is spacelike and N is timelike,

i) In the case of $k_{1g}^2 < a_1^2$

$$\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right| = -k_{1g}^2 + a_1^2 \tag{2.19}$$

Substituting (2.8), (2.9), (2.12), (2.13), (2.15), (2.16)and (2.19) in (2.14), $K'(0)$ can be written as

$$\begin{aligned} K'(0) = & \int\limits_0^l \mu\{-2k''_{1g} - 2a_1a'_2 - 4a_2a'_1 + 2k_{2g}a_1a_3 \\ & +k_{1g}\left(-k_{1g}^2(l) + a_1^2(l) - k_{1g}^2 + a_1^2 + 2k_{2g}^2 - 2a_2^2\right)\}ds \\ & -2\mu'(l)k_{1g}(l) + 2\mu(l)k'_{1g}(l) + 4\mu(l)a_1(l)a_2(l). \end{aligned}$$

In order that $K'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given timelike arc α must satisfy two boundary conditions and differential equation

$$\begin{aligned} &(1) \quad k_{1g}(l) = 0 \\ &(2) \quad k'_{1g}(l) = -2a_1(l)a_2(l) \\ &(3) \quad -2k''_{1g} - 2a_1a'_2 - 4a_2a'_1 + 2k_{2g}a_1a_3 \\ &\qquad +k_{1g}\left(a_1^2(l) - k_{1g}^2 + a_1^2 + 2k_{2g}^2 - 2a_2^2\right) = 0. \end{aligned} \tag{2.20}$$

ii) In the case of $k_{1g}^2 > a_1^2$

$$\left|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2\right| = k_{1g}^2 - a_1^2. \tag{2.21}$$

Substituting (2.8), (2.9), (2.12), (2.13), (2.15), (2.16)and (2.21) in (2.14), $K'(0)$ can be written as

$$\begin{aligned} K'(0) = & \int\limits_0^l \mu\{2k''_{1g} + 2a_1a'_2 + 4a_2a'_1 - 2k_{2g}a_1a_3 \\ & +k_{1g}\left(k_{1g}^2(l) - a_1^2(l) + k_{1g}^2 - a_1^2 - 2k_{2g}^2 + 2a_2^2\right)\}ds \\ & +2\mu'(l)k_{1g}(l) - 2\mu(l)k'_{1g}(l) - 4\mu(l)a_1(l)a_2(l). \end{aligned}$$

In order that $K'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given timelike arc α must satisfy two boundary conditions and differential equation

$$\begin{array}{ll} (1) & k_{1g}(l)=0 \\ (2) & k_{1g}'(l)=-2a_1(l)a_2(l) \\ (3) & 2k_{1g}''+2a_1a_2'+4a_2a_1'-2k_{2g}a_1a_3 \\ & +k_{1g}\left(k_{1g}^2(l)-a_1^2(l)+k_{1g}^2-a_1^2-2k_{2g}^2+2a_2^2\right)=0. \end{array} \tag{2.22}$$

3. Applications

Theorem 3.1. An arc of a geodesic on hyperbolic n-space $H^n(r)$ is a relaxed elastic line .

Proof. For a geodesic arc on hyperbolic n-space $H^n(r)$, $k_{1g}=0$ (so $k_{2g}=0$), $a_1^2=c^2=\dfrac{1}{r^2}$ and $a_2=a_3=0$. Therefore (2.20) and (2.22) are satisfied.

Theorem 3.2. In the spacelike hyperplane in $\mathbb{R}_1^n$, an arc is a relaxed elastic line if and only if it lies on a geodesic.

Proof. In the spacelike hyperplane in $\mathbb{R}_1^n$, k_{2g}, a_2, a_3 vanishes for all curves and $a_1=0$. Then the third equation in (2.20) and (2.22) reduces to

$$2k_{1g}''+k_{1g}^3=0. \tag{3.1}$$

With integrating factor k_{1g}', the first integral is

$$\left(k_{1g}'\right)^2+\frac{1}{4}k_{1g}^4=\text{const.}$$

The boundary conditions in(2.20) and (2.22), which reduces to $k_{1g}'(l)=0$, require that the constant be zero. But then we must have $k_{1g}\equiv 0$.

Conversely, any arc of a geodesic in the spacelike hyperplane satisfies (3.1), (2.20) and (2.22), trivially.

Theorem 3.3. On the spacelike hypersurface in $\mathbb{R}_1^n$, an arc of a geodesic is a relaxed elastic line if and only if it satisfies

$$a_1^2a_2=0.$$

Proof. If $k_{1g} \equiv 0$ (so $k_{2g} = 0$), then the third equation in (2.20) and (2.22) reduces to

$$a_1 a_2' + 2a_1' a_2 = 0.$$

The first integral is

$$a_1^2 a_2 = \text{const}$$

and the constant must vanish because of the second equation in (2.20), (2.22). The boundary conditions in (2.20) and (2.22) are trivial.

Theorem 3.4. An arc of a geodesic on a pseudo-hypersphere $S_1^n(r)$ is a relaxed elastic line.

Proof. For a geodesic arc on hyperbolic n-space $S_1^n(r)$, $k_{1g} = 0$ (so $k_{2g} = 0$), $a_1^2 = c^2 = \dfrac{1}{r^2}$ and $a_2 = a_3 = 0$. Therefore (2.18) is satisfied.

Theorem 3.5. In the timelike hyperplane in $\mathbb{R}_1^n$, an arc is a relaxed elastic line if and only if it lies on a geodesic.

Proof. In the timelike hyperplane, k_{2g}, a_2, a_3 vanishes for all curves and $a_1^2 = c^2 = 0$. The third equation in (2.18) reduces to

$$2k_{1g}'' - k_{1g}^3 = 0. \tag{3.2}$$

With integrating factor k_{1g}', the first integral is

$$\left(k_{1g}'\right)^2 - \frac{1}{4}k_{1g}^4 = \text{const.}$$

The boundary conditions in (2.18), which reduces to $k_{1g}'(l) = 0$, require that the constant be zero. But then we must have $k_{1g} \equiv 0$.

Conversely, any arc of a geodesic in the timelike hyperplane satisfies (26) and (20), trivially.

Theorem 3.6. On the timelike hypersurface in $\mathbb{R}_1^n$, an arc of a geodesic is a relaxed elastic line if and only if it satisfies

$$a_1^2 a_2 = 0.$$

Proof. If $k_{1g} \equiv 0$, then the third equation in (2.18) reduces to

$$a_1 a_2' + 2a_1' a_2 = 0.$$

The first integral is

$$a_1^2 a_2 = \text{const}$$

and the constant must vanish because of the second equation in (2.18). The boundary conditions in (2.18) are trivial.

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Received 29.05.2000