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On Torsion-Free Barely Transitive Groups

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Dedicated to Prof. Dr. M.G. Ikeda on his 70^{th} birthday

Abstract

B. Hartley asked the following question: Does there exist a torsion free barely trasitive group? Existence of torsion free simple barely transitive group is also unknown. We answer the latter question negatively in a special case. Moreover we proved the following: Let G be a simple barely transitive group, and H be a stabilizer of a point. If for a non-identity element $x \in G$, $C_G(x)$ is infinite then, $C_G(x)$ cannot contain H.

Key Words: Barely transitive, commensurable.

Introduction

A group is called a barely transitive group if it acts transitively and faithfully on an infinite set and every orbit of every proper subgroup is finite. The structure of locally finite barely transitive groups is studied in [5], [6] and [2]. In general, not much, known about the structure of arbitrary barely transitive groups. It is proved in [2] that for an arbitrary barely transitive group G, if $G \neq G'$, then G is a locally finite group. Here in Proposition 1 we give another sufficient condition for a barely transitive group to be locally finite; the condition is also necessary.

It is well known that the groups constructed by Ol'shanskii in [7] are simple, periodic barely transitive groups. On the other hand, it is proved in [4] that there exists no simple locally finite barely transitive group. Then the following question is of interest : Does there exist a simple torsion free barely transitive group? This is a special case of a question raised by B. Hartley in [3]. Does there exist a torsion free barely transitive group? The answer is still unknown none the less by using similar technigues as in [1] we

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prove that if the centralizer of a non-trivial element is infinite and contains the stabilizer of a point, then G is not simple, namely:

Theorem 1 *Let G be a simple barely transitive group, and H be the stabilizer of a point. If for a non-identity element $x \in G$, $C_G(x)$ is infinite then, $C_G(x)$ cannot contain H.*

Proof: Assume if possible that there exists a non-trivial element x in G such that $C_G(x)$ is infinite and $C_G(x) \geq H$. Let

$$N = \{y \in G : \mid C_G(x) : C_G(x) \cap C_G(y) \mid < \infty \}.$$

It is clear that N is a subgroup of G. We first show N is normal in G. Let $g \in G$ and $y \in N$. Then $\mid C_G(x^g) : C_G(x^g) \cap C_G(y^g) \mid < \infty$. This implies $\mid C_G(x) \cap C_G(x^g) : C_G(x) \cap C_G(x^g) \cap C_G(y^g) \mid < \infty$. Moreover $\mid C_G(x) : C_G(x) \cap C_G(x^g) \mid \leq \mid C_G(x) : H \mid \mid H : H \cap H^g \mid < \infty$. Hence $\mid C_G(x) : C_G(x) \cap C_G(x^g) \cap C_G(y^g) \mid < \infty$. This implies $\mid C_G(x) : C_G(x) \cap C_G(y^g) \mid < \infty$. It follows that $y^g \in N$. Then $N = G$. This implies that for any $y \in G$ the index $\mid C_G(x) : C_G(x) \cap C_G(y) \mid < \infty$. But on the other hand for any $1 \neq y \in G$, $C_G(y)$ is a proper subgroup of G, hence $\mid C_G(y) : C_G(y) \cap H \mid < \infty$. Also $C_G(y) \cap H \leq C_G(y) \cap C_G(x)$, then $\mid C_G(y) : C_G(y) \cap C_G(x) \mid < \infty$. Hence for a fixed element $x \in G$ for every non identity element y in G the subgroups $C_G(x)$ and $C_G(y)$ are commensurable.

Now we show that G is a minimal non FC-group. Indeed, let K be any proper subgroup of G and $y \in K$. Then $\mid K : K \cap H \mid < \infty$ and $\mid C_G(x) : C_G(x) \cap C_G(y) \mid < \infty$. The group $C_G(x) \geq H$ so $\mid H : H \cap C_G(y) \mid < \infty$. Then $\mid K \cap H : K \cap H \cap C_G(y) \mid < \infty$. It follows that $\mid K : C_{K \cap H}(y) \mid < \infty$. This implies $\mid K : C_K(y) \mid < \infty$ for any $y \in K$.

Let $T(G)$ be the set of torsion elements of G. Assume if possible that G is torsion free. By [8] Theorem 4.32 a torsion free FC-group is abelian then we get for any $1 \neq y \in G, C_G(y)$ is an abelian group. Let $x, y \in G$. Then the index $\mid C_G(x) : C_G(x) \cap C_G(y) \mid < \infty$ implies $C_G(x) \cap C_G(y)$ is infinite. Let $1 \neq a \in C_G(x) \cap C_G(y)$. Then $C_G(a)$ is abelian and contains x and y. This implies G is abelian. Hence $T(G) \neq 1$

Let K be any finite subset of $T(G)$. Then $\mid C_G(x) : C_G(x) \cap C_G(K) \mid < \infty$ by the first paragraph. Since $C_G(x)$ is infinite we get $C_G(x) \cap C_G(K)$ is an infinite subgroup of G. Let $1 \neq a \in C_G(x) \cap C_G(K)$. Then $C_G(a) \supseteq K$ and $C_G(a)$ is a proper subgroup of G. This implies $C_G(a)$ is an FC-group. Hence $\langle K \rangle \leq C_G(a)$. By Dietzmann's lemma this implies $\langle K \rangle$ is finite. Hence $T(G)$ is a subgroup of G this implies $T(G) = G$ i.e. G is a locally finite simple barely transitive group. But this is impossible by [4].

Proposition 1 *A barely transitive group G is a union of an increasing sequence of proper normal subgroups of G if and only if G is locally finite.*

Proof. Assume that G is a union of an increasing sequence of proper normal subgroups N_i. Let H be the stabilizer of a point. Then $|N_i : N_i \cap H| < \infty$. Hence there exists a normal subgroup M of N_i contained in $N_i \cap H$ and of finite index, say m, in N_i. This implies $N_i^m \lhd G$. It follows that $N_i^m = 1$. Then every proper normal subgroup of G has finite exponent and is residually finite. But by [2] a residually finite group of finite exponent is locally finite. This implies N_i is locally finite for all i. It follows that G is locally finite.

In particular G is a countable p-group by [2, Theorem 2].

The converse is an immediate consequence of the fact that a locally finite barely transitive group is a p-group for some prime p, see [2, Theorem 2].

Proposition 2 *Let G be a torsion free barely transitive group and H be the stabilizer of a point. If there exists a non-identity element $x \in G$ such that $|H : C_H(x)| < \infty$, then G is not simple.*

Proof Assume that G is simple. If $H = 1$, then G is periodic. Hence we may assume that $H \neq 1$.

Let $M = \{g \in G : \ |H : C_H(g)| < \infty \}$. Clearly M is a subgroup of G. In fact M is normal in G. Indeed, for $g \in M$ and $t \in G$, we have $|H : C_H(g)| < \infty$ and $|H : H \cap H^t| < \infty$. Then $|H^t : C_{H^t}(g^t)| < \infty$, moreover $|H \cap H^t : C_{H \cap H^t}(g^t)| < \infty$. Hence $|H : C_H(g^t)| < \infty$ and so M is normal. Since x is a non-trivial element in M we get $M = G$.

Now we show that G is a minimal non-FC-group. Let K be any proper subgroup of G and $t \in K$. Then $|K : K \cap H| < \infty$ and $|H : C_H(t)| < \infty$. Hence $|K \cap H : C_{K\cap H}(t)| < \infty$. This implies $|K : C_K(t)| < \infty$ and so K is an FC-group and G is a minimal non-FC-group. But by [8] a torsion free FC-group is abelian. Hence each proper subgroup of G is abelian.

Let x and y be non-identity elements of G. Then $C_G(x)$ and $C_G(y)$ are proper subgroups of G. Since $M = G$ we get $|H : C_H(x)| < \infty$ and $|H : C_H(y)| < \infty$. Then $|H : C_H(x) \cap C_H(y)| < \infty$. Then $|C_G(x) : C_G(x) \cap H| < \infty$ and $|C_G(y) : C_G(y) \cap H| < \infty$ implies $|C_G(x) : C_H(x)| < \infty$ and $|C_H(x) : C_H(x) \cap C_H(y)| < \infty$. Hence $|C_G(x) : C_H(x) \cap C_H(y)| < \infty$. It follows that $|C_G(x) : C_G(x) \cap C_G(y)| < \infty$. But $C_G(x)$ is a torsion free subgroup containing x. Hence $C_G(x) \cap C_G(y)$ is infinite. Let $1 \neq a \in C_G(x) \cap C_G(y)$. Then $C_G(a)$ contains x and y and by above $C_G(a)$ is an abelian group. Hence x and y commute and so G is an abelian group. This contradicts the assumption that G is a torsion free simple group.

Corollary Let G be a torsion free barely transitive group. If FC-center of H is non-trivial, then G is not simple.

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