On Non-Homogeneous Riemann Boundary Value Problem

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Abstract

In this paper we consider non-homogeneous Riemann boundary value problem with unbounded oscillating coefficients on a class of open rectifiable Jordan curve.

Key Words: Curve, Cauchy type integral, singular integral, Riemann problem.

In [1], homogeneous Riemann boundary value problem was studied when curve \( \gamma \) satisfies the condition \( \theta(\delta) \approx \delta \) and \( G \) is an oscillating function at the end points of the curve. In this work we investigate the non-homogeneous Riemann problem in the same case and we will use terminology and notations introduced in [1].

We need the following class of functions for our future references:

\[
H^{a_k}(\mu_k, \nu_k) + H^{a_2}(\mu_2, \nu_2) = \{ g \in C_\gamma : g = g_1 + g_2, g_k \in C_\gamma, \Omega_{\nu_k}^{a_k}(\xi) = O(\xi^{-\nu_k}), \omega^{a_k}_k(\delta, \xi) = O(\delta^{a_k} \xi^{-\mu_k - \nu_k}) \}
\]

where \( k = 1, 2, \mu_k \in (0, 1], \nu_k \in [0, 1), \delta, \xi \in (0, d], \delta \leq \xi, d = \text{diam} \gamma \).

Lemma 1. [3] Suppose that \( \gamma \) satisfies \( \theta(\delta) \approx \delta \), \( G(t) = \exp(2\pi i f(t)) \), \( \Omega^{a_k}_k(\xi) = O(\ln^{1/2} \delta), \omega^{a_k}_k(\delta, \xi) = O(\delta^{1/2}), \delta \leq \xi, k = 1, 2, g \in H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2) \) and suppose

1991 AMS subject classification: Primary 30E20, 30E25; secondary 45E05
h is holomorphic in $\mathbb{C}\setminus\gamma$, continuously extendable to $\hat{\gamma}$ from both sides, $h(z) \neq 0$ for all $z \in \mathbb{C}\setminus\gamma$, $h^\pm(t) \neq 0$ for all $t \in \hat{\gamma}$, $h^+(t) = G(t)h^-(t)$, $g/h^+ \in L(\gamma)$. Then the function

$$\Phi_0(z) = \frac{h(t)}{2\pi i} \int_\gamma \frac{g(\tau)}{h^+(\tau)(\tau - z)} d\tau, z \not\in \gamma$$

(1)

is holomorphic in $\mathbb{C}\setminus\gamma$, continuously extendable to $\hat{\gamma}$ from both sides and satisfies the homogeneous boundary conditions.

We introduce the quantity

$$\Gamma_G^k = \lim_{z \to a_k} \frac{1}{\ln |z - a_k|} Re \int_\gamma \frac{f(\tau)}{\tau - z} d\tau, z \not\in \gamma$$

(2)

and if $\Gamma_G^k$ is finite introduce

$$a_{k'} = \begin{cases} \Gamma_G^k, & \text{if } \Gamma_G^k \in \mathbb{Z} \\ [\Gamma_G^k] + 1, & \text{if } \Gamma_G^k \not\in \mathbb{Z} \end{cases}$$

(3)

$k=1,2$.

**Lemma 2.** Suppose that $\gamma$ satisfies $\theta(\delta) \approx \delta$, $G$ and $g$ are as in lemma and

$$\chi_0(z) = (z - a_1)^{-a_{1'}}(z - a_2)^{-a_{2'}} \exp \int_\gamma \frac{f(\tau)}{\tau - z} d\tau, z \not\in \gamma.$$  

Then $g/\chi_0^+$ is integrable on $\gamma$.

**Proof.** It is obvious that $g/\chi_0^+$ is bounded on a compact subset of $\gamma\setminus\{a_1, a_2\}$ and measurable. Therefore it is integrable on compact subset of $\gamma$. Now we estimate $g/\chi_0^+$ on $\gamma_3$ $(a_1)$ for small enough $\delta$. Since $g \in H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2)$ we have

$$|g(t)| \leq \Omega_2^{a_1}(|t - a_1|) + \Omega_2^{a_2}(|t - a_2|) \leq$$

$$\Omega_2^{a_1}(|t - a_1|) + \Omega_2^{a_2}(|a_2 - a_1| - \delta) \leq C \cdot |t - a_1|^{-\nu_1} + C \cdot |t - a_1|^{-\nu_1}.$$  

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From (2) we have

$$-\text{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau \leq -(|\Delta_G^1| + \varepsilon) \ln |z - a_1|,$$

where $z \not\in \gamma$ is close enough to $a_1$. Hence

$$\left| \frac{1}{\chi_0^\pm(t)} \right| = |(t - a_1)^{\alpha_1'}(t - a_2)^{\alpha_2'}| \exp \left( -\text{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau \right) \leq$$

$$\leq C |(t - a_1)^{\alpha_1'} \exp \left( -(|\Delta_G^1| + \varepsilon) \ln |z - a_1| \right) = C |t - a_1|^{|\alpha_1'-\Delta_G^1+\varepsilon|}.$$

Therefore

$$\left| \frac{g(t)}{\chi_0^\pm(t)} \right| \leq C |t - a_1|^{-\nu_1+\alpha_1'-\Delta_G^1+\varepsilon}.$$

For small enough $\varepsilon$, we have $q = \nu_1 - \alpha_1' + \Delta_G^1 + \varepsilon > -1$, that is, in the vicinity of $a_1$

$$\left| \frac{g(t)}{\chi_0^+(t)} \right| \leq C |t - a_1|^{|q|}, q > -1.$$

Analogously, similar estimation exists in the vicinity of $a_2$. This yields $\frac{g(t)}{\chi_0^-(t)}$ as integrable.

**Lemma 3.** Suppose $\gamma$ satisfies $\theta(\delta) \approx \delta$, $G$ satisfies the conditions of lemma 1, $g \in H^{\alpha_1}(\mu_1, \nu_1) + H^{\alpha_2}(\mu_2, \nu_2)$ and

$$\Delta_G^k = \lim_{z \to a_k} \frac{1}{\ln |z - a_k|} \text{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau, z \not\in \gamma \tag{4}$$

exists. Then the function in (1) is piecewise holomorphic while $h = \chi$.

Proof. It is obvious that according to condition (4) we may take $\alpha_1 = \alpha_1', \alpha_2 = \alpha_2'$ and $\chi_0 = \chi$. Then for function (1) we only need to estimate in endpoints.

We shall investigate function (1) in the vicinity of $a_1$ (in the other end we may show the proof analogously). Take $2\eta = |z - a_1|, q = -\alpha_1 + \Delta_G^1$. Since $q \in (-1, 0], \nu_1 \in [0, 1)$.
Choose $\varepsilon$ small enough such that $q + \nu_1 + \varepsilon < 1$. Let $\lambda > 0$ such that for $z \in \{\xi \in \mathbb{C} : |\xi - u_1| < \lambda\} \setminus \gamma$,

$$(\Delta_G^1 + \varepsilon) \ln |z - a_1| \leq \text{Re} \int \frac{f(\tau)}{\tau - z} d\tau \leq (\Delta_G^1 - \varepsilon) \ln |z - a_1|.$$  \hspace{1cm} (5)

Let $t_z \in \{y \in \gamma : |z - y| = \text{dist}(z, \gamma\lambda(a_1) \setminus \gamma_\eta(z))\}$. We decompose (1) as follows:

$$\frac{\Phi_0(z)}{\chi(z)} = \frac{1}{2\pi i} \int_\gamma \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau = \frac{1}{2\pi i} \int_{\gamma \setminus \gamma_\lambda(a_1)} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau + \frac{1}{2\pi i} \int_{\gamma_\lambda(a_1) \setminus \gamma_\eta(z)} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau + \frac{1}{2\pi i} \int_{\gamma_\eta(z)} \frac{g(\tau) - g(t_z)}{\chi^+(\tau)(\tau - z)} d\tau + \frac{1}{2\pi i} \int_{\gamma_\eta(z)} \frac{g(t_z)}{\chi^+(\tau)(\tau - z)} d\tau$$

$$= A_1 + A_2 + A_3 + A_4.$$  

It is obvious that since $A_1$ does not depend on $\eta$ it is bounded in the vicinity of $a_1$.

Let $\tau \in \gamma \lambda(a_1) \setminus \gamma_\eta(z)$ therefore $|\tau - a_1| + \eta \leq |\tau - z| + 3\eta \leq |\tau - z|$. From lemma 2 and [2] we have

$$|A_2| \leq \frac{1}{2\pi} \int_{\gamma \lambda(a_1) \setminus \gamma_\eta(z)} \frac{\Omega_{\eta_1}(|\tau - a_1|) + \Omega_{\eta_2}(|a_2 - a_1| - \lambda)}{|\chi^+(\tau)| |(\tau - a_1| + \eta)} |d\tau|$$

$$\leq C \int_{\gamma \lambda(a_1)} \frac{|\tau - a_1|^{-\nu_1}}{|\tau - a_1|^{\eta + \varepsilon}} d\tau \leq C \int_0^\lambda x^{-\nu_1 - q - \varepsilon} x + \eta d\theta(x)$$

$$\leq C \int_0^\lambda \frac{x^{-\nu_1 - q - \varepsilon}}{x + \eta} dx \leq C \int_0^\lambda \frac{x^{-\nu_1 - q - \varepsilon}}{\eta + \varepsilon} dx$$

$$\leq C \int_0^\lambda \frac{x^{-\nu_1 - q - \varepsilon}}{\eta} dx + \int_\eta^\lambda x^{-\nu_1 - q - \varepsilon - 1} dx \leq C \eta^{-\nu_1 - q - \varepsilon}.$$  

If $\gamma_\eta(z) = \emptyset$, then $A_3 = 0$. Otherwise, since $|z - t_z| \leq \eta$ and $\gamma_\eta(z) \subset \gamma\eta(t_z)$ we get

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\[
| A_3 | \leq \frac{1}{2\pi} \int_{\gamma_n(z)} \frac{\omega_{a_1}^z(\tau - t_z \mid \eta/2, a_2 - a_1 \mid -\lambda)}{\chi^+(\tau) \mid \tau - t_z \mid} \mid d\tau | \\
\leq \frac{1}{2\pi} \int_{\gamma_n(z)} \frac{|\tau - t_z \mid |^{\mu_2} \eta^{-\mu_1 - \nu_1 + |\tau - t_z \mid |^{\mu_2} \eta^{-\mu_1 - \nu_1}}}{\eta^{p+\nu} \mid \tau - t_z \mid} | d\tau |
\]
\[
\leq C \eta^{-q - \varepsilon} (\tau^{-\mu_1 - \nu_1} \int_{\gamma_n(t_z)} |\tau - t_z \mid ^{\mu_1 - 1} | d\tau |)
\]
\[
+ \int_{\gamma_n(t_z)} |\tau - t_z \mid ^{\mu_2 - 1} | d\tau | \leq C \eta^{-q - \varepsilon} (\tau^{-\mu_1 - \nu_1} \int_{0}^{2\eta} \tau^{\mu_1 - 1} d\theta(\tau))
\]
\[
+ \int_{0}^{2\eta} \tau^{\mu_2 - 1} d\theta(\tau)) \leq C \eta^{-q - \varepsilon} (\tau^{-\mu_1 - \nu_1} \int_{0}^{2\eta} \tau^{\mu_1 - 1} d\tau)
\]
\[
+ \int_{0}^{2\eta} \tau^{\mu_2 - 1} d\theta(\tau) \leq C \eta^{-q - \varepsilon} (\tau^{-\nu_1} + C \eta^{\mu_2}) \leq C \eta^{-q - \varepsilon - \nu_1}.
\]

Now we investigate $A_4$. If $\gamma_n(z) = \emptyset$, then $A_4 = 0$. Otherwise since $|z - t_z| \leq \eta$ and $|\tau_z - a_1| \geq \eta$ then
\[
|g(t_z)| \leq \Omega_{a_1}^{a_2}(|t_z - a_1|) + \Omega_{a_2}^{a_2}(|t_z - a_2|) \leq \Omega_{a_1}^{a_2}(|t_z - a_1|)
\]
\[
+ \Omega_{a_2}^{a_2}(|a_2 - a_1| - \delta) \leq C |t_z - a_1|^{-\nu_1} + C \leq C \eta^{-\nu_2}.
\]

Suppose that $a_2 \notin \gamma_n(z)$ and $\gamma_n(z) = \Lambda \cup (\bigcup_{k=1}^{p} c_k \tilde{d}_k), 1 \leq p \leq \infty, c_k, d_k \in \sum_{\eta}(z) = \{\xi \in \mathbb{C} : |\xi - z| = \eta\}$. Arcs $c_k \tilde{d}_k$ are connected components of $\gamma_n(z)$ and $\Lambda \subset \sum_{\eta}(z)$. The number of $c_k \tilde{d}_k$'s may not be more than countable since arbitrary partition of interval $[0,d], d=\text{diam } \gamma$, is countable.

The points $c_k, d_k, c_k \neq d_k$ divide $\sum_{\eta}(z)$ into two arcs with endpoints $c_k, d_k$. Denote one of them by $\Lambda_k$ oriented from $c_k$ to $d_k$. Let $D$ be the domain bounded by $\Lambda_k \cup c_k \tilde{d}_k$
and $zgD$. If $\text{meas } \Lambda_k \leq \pi \eta$ then $\text{meas } \Lambda_k \leq |c_k - d_k| \frac{\pi}{2} \leq (\text{meas } c_k d_k) \frac{\pi}{2}$. If $\text{meas } \Lambda_k > \pi \eta$ then $c_k d_k \geq 2 \eta \geq \text{meas } \Lambda_k / \pi$. Therefore $\text{meas } \Lambda_k \leq C(\text{meas } c_k d_k)$, $C=\max\{\pi, 2/\pi\} = \pi$. Meanwhile if $\tau \in \sum_\eta(z)$ we have $|\tau - z| = \eta$. By means of Cauchy theorem we get

$$\left| \int_{\gamma_\eta(z)} \frac{d\tau}{\tau - z} \right| = \left| (\int + \sum_{k=1}^p \int_{c_k d_k} \frac{d\tau}{\tau - z}) \right| \leq \left| (\int - \sum_{k=1}^p \int_{c_k d_k} \frac{d\tau}{\tau - z}) \right|$$

$$\leq \frac{1}{\eta} (\text{meas } \Lambda + \sum_{k=1}^p (\text{meas } \Lambda_k))$$

$$\leq \frac{\pi}{\eta} (\text{meas } \Lambda + \sum_{k=1}^p \text{meas } c_k d_k) = \frac{\pi}{\eta} (\text{meas } \Lambda + \text{meas } \bigcup_{k=1}^p c_k d_k) = \frac{\pi}{\eta} \text{meas } \gamma_\eta(z)$$

$$\leq \frac{\pi}{\eta} \text{meas } \gamma_\eta(t_z) = \frac{\pi}{\eta} \theta_\eta(2\eta) \leq \frac{\pi}{\eta} \theta(2\eta) \leq \frac{\pi}{\eta} C \eta = \pi C.$$ 

Therefore $|A_4| \leq \frac{C \eta^{-\nu_1}}{2\pi i} \eta^{-\varepsilon} C \pi = C \eta^{-q-\nu_1-\varepsilon}$. If we round up the result obtained we have $|\Phi_0(z)| \leq \left| \frac{\chi(z)}{2\pi i} \int \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau \right| \leq C \eta^{-q-\nu_1 - \varepsilon} |\chi(z)|$. From (5) $|\chi(z)| \leq C \eta^q - \varepsilon$, therefore $|\Phi_0(z)| \leq C \eta^{-q-\nu_1 - \varepsilon} |\chi(z)| \leq C \eta^{-\nu_1 - 2\varepsilon}$. Since $\varepsilon$ is small enough we may assume that $\nu_1 + 2\varepsilon < 1$. This proves the lemma.

In [1] the solution of the homogeneous boundary value problem was given as $\chi(z)P_{\alpha-1}(z)$ where $P_{\alpha-1}(z)$ is a polynomial whose degree is not greater than $\alpha - 1$. If $\alpha = 0$ $P_{\alpha-1}(z) \equiv 0$. For $\alpha < 0$ the coefficients of $z^{-1}, z^{-2}, \ldots z^{-\alpha}$ in the expansion

$$\frac{1}{2\pi i} \int \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau = -z^{-1} \frac{1}{2\pi i} \int \frac{g(\tau)}{\chi^+(\tau)} d\tau - z^{-2} \frac{1}{2\pi i} \int \frac{g(\tau)}{\chi^+(\tau)} d\tau - z^{-3} \frac{1}{2\pi i} \int \frac{g(\tau)}{\chi^+(\tau)} d\tau - \ldots$$

must be zero. Thus the following theorem is obtained.

**Theorem.** Suppose the conditions of lemma 1 are satisfied and limit in (4) exists.
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i) If \( \alpha \geq 0 \), the Riemann boundary value problem is solvable in \( K(\gamma) \) unconditionally, the solution is given by

\[
\Phi(z) = \frac{\chi(z)}{2\pi i} \int_{\gamma} \frac{g(t)}{\chi^+(t)(t-z)} \, dt + \chi(z) \, P_{\alpha-1}(z),
\]

where \( P_{\alpha-1}(z) \) is arbitrary polynomial of degree not greater than \( \alpha - 1 \) (\( P_{\alpha-1}(z) \equiv 0 \) for \( \alpha = 0 \)).

ii) If \( \alpha < 0 \), then the Riemann boundary value problem is solvable in \( K(\gamma) \) if and only if the conditions

\[
\int_{\gamma} \frac{g(t)}{\chi^+(t)} \, dt = 0, \quad j = 0, 1, \ldots, -\alpha - 1
\]

are satisfied. Under these conditions the solution is unique and is given by

\[
\Phi(z) = \frac{\chi(z)}{2\pi i} \int_{\gamma} \frac{g(t)}{\chi^+(t)(t-z)} \, dt.
\]

Acknowledgement

The author would like to thank professor R. K. Seifullaev for his valuable comments.

References


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Received 04.02.1999

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