Conjugacy Structure Type and Degree Structure Type in Finite $p$–groups

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Abstract

Let $G$ be a finite $p$–group, and denote by $k(G)$ number of conjugacy classes in $G$. The aim of this paper is to introduce the conjugacy structure type and degree structure type for $p$–groups, and determine these parameters for $p$–groups of order $p^5$, and calculate $k(G)$ for them.

Key Words: breadth, conjugacy structure type, degree structure type.

1. Introduction

Let $G$ be a finite $p$–group, and denote by $k(G)$ number of conjugacy classes of $G$. We remind the reader that an element $g$ of $p$–group $G$ is said to have breadth $b_G(g)$ (or $b(g)$ if no ambiguity is possible) if $p^{b_G(g)}$ is the size of conjugacy class of $g$ in $G$. The breadth $b(G)$ of $G$ will be maximum of breadths of its elements. We have,

$b(G) = 1$ if and only if $|G'| = p$ (see [4]),

$b(G) = 2$ if and only if $|G'| = p^2$ or $|G : Z(G)| = p^3$ and $|G'| = p^3$ (see [7]).

Definition 1. Let $s_i$ be the number of conjugacy classes of size $p^i$ in $G$. Let $m$ be a non-negative integer such that $s_m \neq 0$, and $s_i = 0$ for $i > m$. Then $|G| = \sum_{i=0}^{m} s_ip^i$, and $k(G) = \sum_{i=0}^{m} s_i$. We define the tuple $(s_0, s_1, \ldots, s_m)$, Conjugacy Structure Type of $G$, and denote by $T_c(G)$. It is clear that $G$ is abelian if and only if $m = 0$. 

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Definition 2. Let $\alpha_i$ be the number of irreducible characters of $G$ of order $p^i$. Let $h$ be a non-negative integer such that $\alpha_h \neq 0$, and $\alpha_i = 0$ for $i > h$. Then $|G| = \sum_{i=0}^h \alpha_i p^{2i}$.

We define the tuple $(\alpha_0, \alpha_1, \ldots, \alpha_h)$. Degree Structure Type of $G$, and denote by $T_d(G)$.

We know that $b(G)$ is the maximum index of $i$ such that $s_i$ is nonzero, that means $b(G) = m$. We denote by $\beta(G)$ the maximum index of $i$ such that $\alpha_i$ is nonzero that is $\beta(G) = h$.

Burnside’s Formula. Let $G$ be a finite $p$–group and $M$ be a maximal subgroup in $G$. If $s$ and $t$ are the number respectively of invariant and fused conjugacy classes of $M$ then

\[ k(G) = p^s + \frac{t}{p} + \frac{k(M)}{p}. \]

Proof. See [1,p.472].

The main theorem is:

**Theorem A.** Let $G$ be a nonabelian finite $p$–group of order $p^5$. Then one of the following occurs:

(i) $k(G) = p^4 + p^3 - p^2$, $T_d(G) = (p^4, p^3 - p^2)$,

(ii) $k(G) = p^4 + p - 1$, $T_d(G) = (p^4, 0, p - 1)$,

(iii) $k(G) = p^3 + p^2 - 1$, $T_d(G) = (p^2, p^3 - 1)$ or $(p^3, p^2 - p, p - 1)$,

(iv) $k(G) = 2p^3 - p$, $T_d(G) = (p^3, p^3 - p)$,

(v) $k(G) = 2p^2 + p - 2$, $T_d(G) = (p^2, p^2 - 1, p - 1)$.

2. Elementary Lemmas

Throughout this section, $G$ denote a $p$–group of order $p^n$. To proof the main theorem we need some lemmas:

**Lemma 1.**

(i) Let $G$ be a nonabelian finite $p$–group. If $b(G) \geq k$, then $|G : Z(G)| \geq p^{k+1}$.

(ii) Let $G$ be a nonabelian finite $p$–group. If $\beta(G) \geq 2$, then $|G : Z(G)| \geq p^4$.

**Proof.**

(i) Suppose that $g \in G$ such that $|G : C_G(g)| \geq p^k$. Since $Z(G) \subset C_G(g)$ we
have

$$|G : Z(G)| > |G : C_G(g)| \geq p^k$$

Therefore $|G : Z(G)| \geq p^{k+1}$.

(ii) It is clear from the fact that for any irreducible character $\chi$ of $G$,

$$\chi(1)^2 \leq |G : Z(G)|.$$

\[\Box\]

**Lemma 2.** Let $G$ be a finite $p$-group with $b(G) = 1$ and $\beta(G) = \beta$. Then

(i) $G/Z(G)$ is an elementary abelian subgroup of order $p^{2\beta}$.

(ii) Every character of $G$ has degree 1 or $p$.

(iii) $k(G) = p^{n-1} + p^{n-2\beta} - p^{n-2\beta-1}$,

\[T_d(G) = (p^{n-1}, p^{n-2\beta} - p^{n-2\beta-1}), T_c(G) = (p^{n-2\beta}, p^{n-1} - p^{n-2\beta-1}).\]

**Proof.** We have $|G'| = p$. Hence $G' \subseteq Z(G)$ and $G/Z(G)$ is abelian. We know that exponent of $G/Z(G)$ is $p$ (see [5]). Therefore $G/Z(G)$ is elementary abelian. If $\chi$ is a nonlinear irreducible character of $G$, then

$$\chi(1)^2 = |G : Z(G)|$$

by exercise 2.13 of [3]. Hence $\chi(1) = p^\beta$ for any nonlinear irreducible character $\chi$ of $G$. So by character degrees formula,

$$k(G) = p^{n-1} + p^{n-2\beta} - p^{n-2\beta-1}, T_d(G) = (p^{n-1}, p^{n-2\beta} - p^{n-2\beta-1}).$$

since $p^n = p^s + s_1p$, where $|Z(G)| = p^s$. We have

$$T_c(G) = (p^{n-2\beta}, p^{n-1} - p^{n-2\beta-1}).$$

\[\Box\]
Corollary 1. Let $G$ be a nonabelian $p$–group of order $p^3$. Then $k(G) = p^2 + p - 1$ and $T_d(G) = T_c(G) = (p, p^2 - 1)$.

Proof. It is clear by $\beta(G) = 1$. □

Lemma 3. Let $G$ be a finite $p$–group and $b(G) \geq 2$. If $|G : G'| = p^k$, then $2 \leq k \leq n - 2$.

Proof. By lemma. 1(ii) of [2], $|G'| \geq p^2$ and by character degrees formula proof is trivial. □

Corollary 2. Let $G$ be a nonabelian $p$–group of order $p^4$. Then one of the following occurs:

(i) $k(G) = p^3 + p^2 - p$, $T_c(G) = (p^2, p^3 - p)$, and $T_d(G) = (p^3, p^2 - p)$,

(ii) $k(G) = 2p^2 - 1$, $T_c(G) = (p, p^2 - 1, p^2 - p)$, and $T_d(G) = (p^2, p^2 - 1)$.

Proof. It is clear by lemmas 2 and 3. □

Example 1. Let $G = E(p^3) = \langle x, y | x^p = y^p = [x, y]^p = 1, [x, y] \in Z(G) \rangle$. We know that all of conjugacy classes of order 1 are in $Z(G)$, and has form $\{[x, y]^i\}$ for some $i = 1, 2, \ldots, p$.

Other classes of $G$ are:

- Classes of the form $\{x^i[x, y]^j | 0 \leq j \leq p - 1\}$ where $i = 1, 2, \ldots, p - 1$.

- Classes of the form $\{y^i[x, y]^j | 0 \leq j \leq p - 1\}$ where $i = 1, 2, \ldots, p - 1$.

- Classes of the form $\{x^iy^j[x, y]^k | 0 \leq k \leq p - 1\}$ where $i, j = 1, 2, \ldots, p - 1$.

Hence $T_c(E(p^3)) = (p, p^2 - 1)$. 324
Lemma 4. \( \text{Let } G \text{ be a finite } p\text{-group and } M \text{ be an abelian maximal subgroup of } G. \) Then \( k(G) = p^{n-2} + p^{z+1} + p^{z-1}, \text{ where } |Z(G)| = p^z. \)

**Proof.** We know that \( Z(G) \subseteq M, \) otherwise \( M' = G', \) which is a contradiction. Then the Burnside’s formula completes the proof. \( \square \)

3. **Proof of Theorem A**

In this section we proof theorem A and present some other information about conjugacy structure type:

**Proof.** We consider three possible cases:

**Case 1.** Let \( b(G) = 1. \) Then \( |G'| = p. \) By lemma 2, for \( |G : Z(G)| = p^2 \) or \( p^4 \) we have,

\[
k(G) = p^4 + p^3 - p^2, \quad T_d(G) = (p^4, p^3 - p^2), \text{ and } T_c(G) = (p^3, p^4 - p^2), \text{ or}
\]
\[
k(G) = p^4 + p - 1, \quad T_d(G) = (p^4, 0, p - 1), \text{ and } T_c(G) = (p, p^4 - 1).
\]

**Case 2.** Let \( b(G) = 2. \) Then \( |G'| = p^2 \) or \( |G'| = p^3 \) and \( |G : Z(G)| = p^3. \)

First suppose \( |G : Z(G)| = p^3, \) then by lemma 1(i). For \( |G'| = p^2 \) or \( p^3, \) we have

\[
k(G) = 2p^3 - p, \quad T_d(G) = (p^3, p^3 - p), \text{ and } T_c(G) = (p^2, P^3 - p, p^3 - p^2), \text{ or}
\]
\[
k(G) = p^3 + p^2 - 1, \quad \text{or } T_d(G) = (p^2, p^3 - 1), \text{ and } T_c(G) = (p^2, 0, p^3 - 1).
\]

Now suppose \( |G : Z(G)| = p^4. \) Then \( |G'| = p^2 \) and \( k = 3. \) Since \( \alpha_3p^{2i} \) is divided by \( (p - 1)p^k \) (see corollary 11 of [6]), then by character degrees formula,

\[
p^5 = p^3 + p(p - 1)t_1p^2 + (p - 1)t_2p^4
\]
for some non-negative integer \( t_1 \) and \( t_2. \) Hence \( t_1 = t_2 = 1\) and

\[
k(G) = p^3 + p^2 - 1, \quad T_d(G) = (p^3, p^2 - p, p - 1), \quad T_c(G) = (p, p^2 - 1, p^3 - p).
\]

**Case 3.** Let \( b(G) = 3. \) Then \( |G : Z(G)| = p^4 \) and \( |G'| = p^3, \) by lemma 1. If \( G \) has an abelian maximal subgroup then \( k(G) = p^3 + p^2 - 1\) (by lemma 4), and \( T_d(G) = (p^2, p^3 - 1). \)

If \( \beta(G) = 2, \) then By character degrees formula,

\[
p^5 = p^2 + \alpha_1p^2 + \alpha_2p^4, \text{ which implies that } 1 + \alpha_1 = hp^2 \text{ for some non-negative integer}
\]
h. Hence $\alpha_2 = p - h$. Since $\alpha_2$ is nonzero and divided by $p - 1$ (by corollary 11 of [6]), $h = 1$. Therefore $k(G) = 2p^2 + p - 2$ and $T_d(G) = (p^2, p^2 - 1, p - 1)$.

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References


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