Multipliers between Orlicz Sequence Spaces *

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Abstract

Let $M, N$ be Orlicz functions, and let $D(\ell_M, \ell_N)$ be the space of all diagonal operators (that is multipliers) acting between the Orlicz sequence spaces $\ell_M$ and $\ell_N$. We prove that the space of multipliers $D(\ell_M, \ell_N)$ coincides with (and is isomorphic to) the Orlicz sequence space $\ell_{M \vee N}$, where $M \vee N$ is the Orlicz function defined by $M \vee N(x) = \sup\{N(\lambda x) - M(x), \ x \in (0, 1)\}$.

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Let $M(t), t \geq 0$, be an Orlicz function, that is a non-decreasing convex function such that $M(0) = 0$ and $M(t) \to \infty$ as $t \to \infty$. Orlicz sequence space $\ell_M$ defined by the function $M(t)$ is the linear space of all sequences of scalars $x = (x_i)_i$ such that $\sum_i M(x_i) < \infty$. Equipped with the norm

$$\|x\|_M = \inf\{\rho : \sum_i M(|x_i|/\rho) \leq 1\}$$

it is a Banach space.

An Orlicz function $M(t)$ is said to satisfy the $\Delta_2$-condition near 0 if $M(2t) \leq CM(t), \ t \in (0, 1)$ for some constant $C > 0$. The following facts are known:

Proposition 1 Let $M$ be an Orlicz function. Then the subspace

$$h_M = \{x = (x_i) : \sum M(|x_i|/\rho) < \infty \ \forall \rho > 0\}$$

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is a closed subspace of $\ell_M$, and the vectors $(e_n)_{n=1}^{\infty}$ (where $e_n = (e_{ni})$, $e_{ni} = 0$ if $i \neq n$, and $e_{nn} = 1$) form a basis in it.

Moreover, $h_m = \ell_M$ if and only if $M$ satisfies the $\Delta_2$-condition.

We refer to [1] for a proof of this proposition and the basic theory of Orlicz sequence spaces.

Let $M(t)$ and $N(t)$ be two Orlicz functions. A sequence of scalars $\lambda = (\lambda_i)$ is called a multiplier between the Orlicz spaces $\ell_M$ and $\ell_N$ if for each $x = (x_i) \in \ell_M$ we have $\lambda x := (\lambda_i x_i) \in \ell_N$. It is easy to see by the Closed Graph Theorem that each multiplier $\lambda$ defines a continuous diagonal operator

$$T_\lambda : \ell_M \to \ell_N.$$ 

Therefore we identify multipliers with diagonal operators and denote by $D(\ell_M, \ell_N)$ the space of all multipliers between $\ell_M$ and $\ell_N$. Regarded with the usual operator norm it is a Banach space.

Consider the function

$$M_N^*(s) = \max(0, \sup_{t \in [0,1]} \{N(st) - M(t)\}), \quad s \geq 0. \quad (1)$$

Evidently it is an Orlicz function, and by its definition we have

$$N(ts) \leq M(t) + M_N^*(s), \quad (2)$$

which generalizes the classical Young inequality.

Two Orlicz functions $M(t)$ and $\tilde{M}(t)$ are equivalent, if

$$\exists c > 0, t_0 > 0 : \quad c^{-1} M(e^{-1} t) \leq \tilde{M}(t) \leq c M(ct), \quad \forall t \in [0, t_0].$$

Equivalent Orlicz functions generate one and the same Orlicz sequence space and define equivalent norms on it. It is easy to see that if one replaces the functions $M$ and $N$ with equivalent Orlicz functions $\tilde{M}$ and $\tilde{N}$, then the functions $M_N^*$ and $\tilde{M}_N^*$ will be equivalent.

**Proposition 2** If $\lambda \in \ell_{M_N^*}$ then it is a multiplier from $\ell_M$ into $\ell_N$. Moreover, the following generalization of the Hölder inequality holds:

$$\|\lambda x\|_N \leq 2\|\lambda\|_{M_N^*} \|x\|_M \quad \forall \lambda \in \ell_{M_N^*}, \forall x \in \ell_M. \quad (3)$$
Proof. First, observe that if $S$ is an Orlicz function then

$$\|(y_i)\|_S > 1 \Rightarrow \|(y_i)\|_S \leq \sum_i S(|y_i|).$$

Indeed, since $S$ is a convex function and $S(0) = 0$ we have for every $\beta > 1$ that $S(\beta^{-1} t) = S(\beta^{-1} t + (1 - \beta^{-1}) \cdot 0) \leq \beta^{-1} S(t)$. Therefore, from the definition of the norm $\|(y_i)\|_S$, it follows that for every $\beta$ such that $1 < \beta < \|(y_i)\|_S$ we have

$$1 < \sum_i S(|y_i|/\beta) \leq \beta^{-1} \sum_i S(|y_i|).$$

So, letting $\beta \to \|(y_i)\|_S$ we obtain $\|(y_i)\|_S \leq \sum_i S(|y_i|)$.

Fix $\lambda = (\lambda_i) \in \ell_{M^N}$ and $x = (x_i) \in \ell_M$, and let

$$\rho > \|\lambda\|_{M^N}, \quad r > \|x\|_M.$$

Consider the sequences $\tilde{\lambda} = \lambda/\rho$ and $\tilde{x} = x/r$.

Then

$$\sum_i N(|\tilde{\lambda}_i\tilde{x}_i|) \leq \sum_i M(|\tilde{x}_i|) + \sum_i M^*_N(|\tilde{\lambda}_i|) \leq 2,$$

and from (4) it follows that $\|\tilde{\lambda}\|_N \leq 2$, thus $\|\lambda x\|_N \leq 2\rho r$. Letting $\rho \uparrow \|\lambda\|_{M^N}$ and $r \uparrow \|x\|_M$ we obtain the claim.

Theorem 3 For every pair of Orlicz functions $M, N$ the sequence spaces $D(\ell_M, \ell_N)$ and $\ell_{M^N}$ coincide as sets, and moreover, they are isomorphic as Banach spaces.

Proof. First, observe that if $S$ is an Orlicz function then

$$\|(y_i)\|_S < 1 \Rightarrow \sum_i S(|y_i|) \leq \|(y_i)\|_S.$$ 

Indeed, since $S$ is a convex function and $S(0) = 0$ we have for $\alpha \in (0, 1)$ that $S(\alpha t) \leq \alpha S(t)$. Therefore for every $\alpha$ such that $\|(y_i)\|_S < \alpha < 1$

$$\|(y_i)\|_S < \alpha \Rightarrow \sum_i S(|y_i|/\alpha) \leq 1 \Rightarrow \sum_i S(|y_i|) \leq \alpha \sum S(|y_i|/\alpha) \leq \alpha,$$

so letting $\alpha \to \|(y_i)\|_S$ we obtain $\sum N(|y_i|) \leq \|(y_i)\|_S$. 

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Consider, in the space of multipliers $D(\ell_M, \ell_N)$, the operator norm

$$\|\lambda\|_0 = \sup \{ \|\lambda x\|_N : \|x\|_M = 1 \}.$$  

From Proposition 2 it follows immediately that

$$D(\ell_M, \ell_N) \supset \ell_{M_N}^*$$

and

$$\|\mu\|_0 \leq 2\|\mu\|_{M_N^*} \quad \forall \mu \in \ell_{M_N^*}.$$  

Further we show that

$$D(\ell_M, \ell_N) \subset \ell_{M_N^*}$$

and

$$\|\mu\|_{M_N^*} \leq 2\|\mu\|_0 \quad \forall \mu \in D(\ell_M, \ell_N).$$  

We may assume without loss of generality that $M(1) = 1$ and $N(1) = 1$. Then we have

$$\forall i \quad \|e_i\|_M = 1, \quad \|e_i\|_N = 1.$$  

Fix a multiplier $\lambda = (\lambda_i) \in D(\ell_M, \ell_N)$ such that $\|\lambda\|_0 = 1/2$. Then $|\lambda_i| = \|\lambda e_i\|_N \leq 1/2 \|e_i\|_M = 1/2$.

Since $M$ and $N$ are Orlicz functions they are continuous. Thus for every $i = 1, 2, \ldots$ there exists an $x_i \in [0, 1]$ such that

$$M_N^*(|\lambda_i|) = N(|\lambda_i|x_i) - M(x_i),$$

that is

$$N(|\lambda_i|x_i) = M(x_i) + M_N^*(|\lambda_i|). \quad (6)$$

Consider the sequence $(x_i)^\infty$. Since by our assumption $\|\lambda\|_0 = 1/2$, we have by (5)

$$\forall i \quad N(|\lambda_i|x_i) \leq \|\lambda_i x_i e_i\|_N \leq 1/2 \|x_i e_i\|_M \leq 1/2,$$

therefore

$$M(x_i) = N(|\lambda_i|x_i) - M_N^*(x_i) < 1/2, \quad i = 1, 2, \ldots. \quad (7)$$
We shall prove by induction that $\sum_1^n M(x_i) \leq 1/2$. It was shown that the statement is true for $n = 1$.

Consider the sequences $\xi^{(n)} = \sum_1^n x_i e_i$, $n = 1, 2, \ldots$. Assume that the claim is true for some $n$. Then
\[
\sum_{i=1}^{n+1} M(x_i) = \sum_{i=1}^n M(x_i) + M(x_{n+1}) \leq 1/2 + 1/2 \leq 1,
\]
so $\|\xi^{n+1}\|_M \leq 1$. Therefore we obtain by (6) and (5)
\[
\sum_{i=1}^{n+1} M(x_i) \leq \sum_{i=1}^{n+1} N(|\lambda_i| x_i) \leq \|\lambda\|_N \leq 1/2,
\]
which proves the claim.

Since $\sum_1^n M(x_i) < 1/2$ for every $n$ we have $\sum_1^\infty M(x_i) \leq 1/2$, thus $x \in \ell_M$ and $\|x\|_M < 1$. Now from (6) and (5) it follows
\[
\sum_{i=1}^\infty M_N(\lambda_i) \leq \sum_{i=1}^\infty N(|\lambda_i| x_i) \leq \|\lambda\|_N \leq 1/2 \|x\|_M \leq 1/2,
\]
and hence $\lambda \in \ell_{M_N}$ and $\|\lambda\|_{M_N} \leq 1$.

Suppose $\mu \in D(\ell_M, \ell_N)$ is an arbitrary multiplier. Consider the sequence $\lambda = \mu/\rho$, where $\rho = 2\|\mu\|_0$. Then we have $\lambda \in \ell_{M_N}$ and $\|\lambda\|_{M_N} = \|\mu/\rho\|_{M_N} \leq 1$, hence $\mu \in \ell_{M_N}$ and
\[
\|\mu\|_{M_N} \leq 2\|\mu\|_0.
\]
The theorem is proved.

Remark 1. An Orlicz function $S$ is called degenerate, if $S(t) = 0$ for some $t > 0$; then the corresponding Orlicz sequence space $\ell_S$ coincides with $\ell_\infty$. In view of the theorem $D(\ell_M, \ell_N) = \ell_\infty$ if and only if the Orlicz function $M_N$ is degenerate.

Example. It is well known that for $p, q \geq 1$
\[
D(\ell_p, \ell_q) = \begin{cases} \ell_r, & 1/r = 1/q - 1/p, \quad \text{if } p > q; \\ \ell_\infty, & \text{if } p \leq q. \end{cases}
\]
Let us see how this result follows from Theorem 2. Consider $M(t) = t^p/p$ and $N(t) = t^q/q$. If $p > q$ then it is easy to see that for each fixed $s \in (0, 1)$ the expression $N(st) - M(t) = (st)^q/q - t^p/p$ attains its maximum for $t \in [0, 1]$ at $t = s^{q/(p-q)}$. Thus for $s \in [0, 1]$

$$M_N^*(s) = \left(1/q - 1/p\right) s^{q/(p-q)} = s^{r}/r$$

with $1/r = 1/q - 1/p$, hence $D(\ell_q, \ell_p) = \ell_r$. In the case $p \leq q$, if $s^q \leq q/p$, then

$$N(st) - M(t) = (st)^q/q - t^p/p \leq 0, \quad t \in [0, 1].$$

Therefore $M_N^*(s) = 0$ for $s \leq (q/p)^{1/q}$, that is $M_N^*$ is a degenerate Orlicz function, hence $D(\ell_p, \ell_q) = \ell_\infty$.

Remark 2. Let $D_c(\ell_M, \ell_N)$ be the space of all compact multipliers between the spaces $\ell_M$ and $\ell_N$. It is easy to see by Proposition 1 that each multiplier from the subspace $h_{M_N}$ is compact (as limit of finitely-supported multipliers), thus

$$h_{M_N} \subset D_c(\ell_M, \ell_N).$$

In particular, if the function $M_N^*$ satisfies the $\Delta_2$-condition near zero, then each multiplier between the spaces $\ell_M$ and $\ell_N$ is compact, that is

$$D(\ell_M, \ell_N) = D_c(\ell_M, \ell_N).$$

Up to our knowledge the following question is open:

Question. Is it true that every compact multiplier between the spaces $\ell_M$ and $\ell_N$ is a limit of finitely-supported multipliers?

Obviously, a positive answer to that question would imply that

$$D_c(\ell_M, \ell_N) = h_{M_N^*},$$

so we would have

$$D(\ell_M, \ell_N) = D_c(\ell_M, \ell_N).$$

if and only if the function $M_N^*$ satisfies the $\Delta_2$-condition.
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