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On Generalized Higher Derivations*

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Abstract

We define the notion of generalized higher derivations and give some elementary relations between generalized higher derivations and higher derivations in the usual sense. We extend the result of an exact sequence of the set of all derivations $\mathrm{Der}(A, M)$ and the set of all generalized derivations $g\mathrm{Der}(A, M)$ given in [N1, Theorem 2.4]. Moreover, we discuss generalized higher Jordan derivations and Lie derivations.

0. Introduction

The notion of generalized derivations on a ring A which was introduced by M. Brešar [B] is related to a derivation of A. In [N1], the author defined anothor type of generalized derivations without using derivations, and give some categorical properties of that generalized derivations. When A has an identity element, these two notions coincide. The results in [N1] were extended to generalized Jordan and Lie derivations in [N2].

On the other hand, in his paper [R], P. Ribenboim gave some properties of higher derivations of modules. His higher derivation f from an A-module M to M is defined by using a higher derivation $d = (d_t) : A \to A$ and the case of length 1 is nothing but a generalized derivation in the sense of Brešar whenever d_0 is the identity map on A.

In this note, we define a generalized higher derivation without using a higher derivation at the viewpoint of [N1] and give some categorical properties which are related to [N1]. We also treat generalized higher Jordan and Lie derivations.

In §1, we give definitions and elementary properties of generalized derivations, generalized Jordan and Lie derivations which were given in [N1, N2] for convenience to the reader. In §2, we define generalized higher derivations and give a relation between higher derivations and generalized higher derivations. Moreover, we discuss the relations

*Dedicated to the memory of my friend, Professor Mehmet Sapancı of higher Jordan and Lie derivations and their generalizations. In §3, we consider the generalized higher derivations of length 2. Then we have an exact sequence which is a natural extension of the length 1 in [N1, Theorem 2.4]. In §4, we discuss higher Jordan derivations. Since the properties of Jordan derivations are similar to derivations, we have an exact sequence which was given in [R, §2]. Higher Lie derivations are treated in the final section 5. In the stand point of ring theory, it seems that Jordan derivations and Lie derivation are quite different properties.

In the following, A and B are k-algebras over a commutative ring k. A left A/k-module M means that M is a left A-module such that $a(\alpha\omega) = \alpha(a\omega)$ and $\alpha\omega = \omega\alpha$ for any $a \in A$, $\alpha \in k$, $\omega \in M$. An $(A/k, B/k)$-bimodule M means a left A/k and a right B/k-module such that $a(\omega b) = (a\omega)b$ ($a \in A$, $b \in B$, $\omega \in M$). In this case, if $A = B$, then we say M an A/k-bimodule. All maps are k-linear maps unless otherwise stated.

1. Preliminaries

In this section, we give definitions of some type of derivations. Let M be an A/k-bimodule, $d : A \to M$ a k-linear map and x, $y \in A$. d is called a *derivation* if

$$d(xy) = d(x)y + xd(y). \tag{1.1}$$

$J : A \to M$ is called a *Jordan derivation* if

$$J(x^2) = J(x)x + xJ(x), \tag{1.2}$$

and $L : A \to M$ is called a *Lie derivation* if

$$L([x, y]) = [L(x), y] + [x, L(y)], \tag{1.3}$$

where $[x, y] = xy - yx$. The properties of these derivations were discussed in a lot of papers till now. In [B], Brešar defined the notion of generalized derivation as follows. $f : A \to M$ is called a *generalized derivation* if there exists a derivation $d : A \to M$ such that

$$f(xy) = f(x)y + xd(y). \tag{1.4}$$

We call f a *d-derivation*. Using this idea, we can define a generalized Jordan and Lie derivations in the sense of Brešar as follows: $g : A \to M$ is called a *J-Jordan derivation* if

$$g(x^2) = g(x)x + xJ(x), \tag{1.5}$$

and $h : A \to M$ is called a *L-Lie derivation* if

$$h([x, y]) = [h(x), y] + [x, L(y)], \tag{1.6}$$

where J (resp. L) is a Jordan (resp. Lie) derivation. We define them without using the corresponding derivations as follows: Let $f,\ g,\ h : A \to M$ be k-linear maps and ω an element of M. A pair $(f\ ;\ \omega)$ is called a *generalized derivation* if

$$f(xy) = f(x)y + xf(y) + x\omega y. \tag{1.7}$$

And if

$$g(x^2) = g(x)x + xg(x) + x\omega x, \tag{1.8}$$

then $(g\ ;\ \omega)$ is called a *generalized Jordan derivation*, and if

$$h([x,y]) = [h(x),y] + [x,f(y)] + [x,\omega,y], \tag{1.9}$$

then $(h\ ;\ \omega)$ is called a *generalized Lie derivation*, where $[x,\omega,y] = x\omega y - y\omega x$. These definitions are given in [N1] and [N2].

The relation between these derivations and generalized them are as follows:

Lemma 1.1([N1, Lemmas 2.1 and 2.2], [N2, Lemmas 2.1 and 2.5]). *Let M be an A/k-bimodule and $f : A \to M$ a k-linear map. For $\omega \in M$, ω_ℓ is a left multiplication of ω. Then the following holds.*

(1a) *If $(f\ ;\ \omega)$ is a generalized derivation, then $f + \omega_\ell$ is a derivation and f is a $(f + \omega_\ell)$-derivation.*

(1b) *If f is a derivation, then $(f + \omega_\ell\ ;\ -\omega)$ is a generalized derivation.*

(2a) *If $(f\ ;\ \omega)$ is a generalized Jordan derivation, then $f + \omega_\ell$ is a Jordan derivation and f is a $(f + \omega_\ell)$-Jordan derivation.*

(2b) *If f is a Jordan derivation, then $(f + \omega_\ell\ ;\ -\omega)$ is a generalized Jordan derivation.*

(3a) *If $(f\ ;\ \omega)$ is a generalized Lie derivation, then $f + \omega_\ell$ is a Lie derivation. In this case, if ω is contained in* $C(M) = \{\tau \in M \mid \tau x = x\tau \text{ for any } x \in A\}$, *then f is a $(f + \omega_\ell)$-Lie derivation.*

(3b) *If f is a Lie derivation, then $(f + \omega_\ell\ ;\ -\omega)$ is a generalized Lie derivation.*

Remark 1.2. The corresponding Lemma also holds for a right multiplication ω_r. For a d-derivation f, if A has an identity element 1, then $(f\ ;\ -f(1))$ is our generalized derivation. But for a J-Jordan derivation (resp. L-Lie derivation) g, it is not known that how can we construct our generalized Jordan derivation (resp. Lie derivation) from g.

An important relation of derivations and generalized derivations is given by the following split exact sequence of k-modules ([N1,Theorem 2.4]).

$$0 \longrightarrow M \xrightarrow{\psi_M} g\mathrm{Der}(A,M) \xrightarrow{\varphi_M} \mathrm{Der}(A,M) \longrightarrow 0 \tag{1.10}$$

where $g\mathrm{Der}(A,M)$ (resp. $\mathrm{Der}(A,M)$) is the set of all generalized derivations (resp. derivations) from A to M, $\psi_M(\omega) = (\omega_\ell\ ;\ -\omega)$ and $\varphi_M(f\ ;\ \omega) = f + \omega_\ell$. The similar exact sequences of generalized Jordan and Lie derivations were given in [N2, Theorems 2.3 and 2.6].

2. Generalized higher derivations and their deformations

In this section, we give definitions of higher derivations and Ribenboim's d-derivations. After that we define generalized higher derivations without using higher derivations and give the relation of higher derivations and generalized higher derivations.

Let $I = \{0, 1, \cdots, n\}$ or $I = \boldsymbol{N} = \{0, 1, 2, \cdots\}$ (with $n = \infty$ in this case). Let B be an A/k-bimodule and let $d_t : A \to B$ be k-linear maps ($t \in I$). $d = (d_t)$ is called a *higher derivation of length* n from A to B if for any $t \in I$ and $x, y \in A$, there holds

$$d_t(xy) = \sum_{i=0}^{t} d_i(x)d_{t-i}(y). \tag{2.1}$$

Then d_0 is a ring homomorphism and d_1 is a (d_0, d_0)-derivation in the usual sense. We denote $\mathrm{Der}^n(A, B)$ the set of all higher derivations of length n from A to B.

In [R], Ribenboim defined a higher derivation from a right A/k-module M to a right B/k-module N as follows. Let $f_t : M \to N$ be k-linear maps and $d = (d_t) \in \mathrm{Der}^n(A, B)$. $f = (f_t)$ is called a *higher d-derivation of length* n from M to N if for any $t \in I$ and $x \in A$, $\omega \in M$, there holds

$$f_t(\omega x) = \sum_{i=0}^{t} f_i(\omega)d_{t-i}(x). \tag{2.2}$$

If $d = (\iota_A, d_1)$ is a higher derivation of length 1 from A to A, then for a higher d-derivation $f = (\iota_A, f_1)$ of length 1, f_1 is the Beršar's generalized derivation, where $\iota_A : A \to A$ is the identity map. The set of all higher d-derivations of length n from M to N is denoted by d-$\mathrm{Der}^n(M, N)$.

Now we define a generalized higher derivation at the stand point of (1.7).

Definition 2.1. Let B be an A/k-bimodule. For k-linear maps $f_t : A \to B$ and $b_t \in B$ ($t \in I$), $f = (f_0, f_1, \cdots, f_n\ ;\ b_0, b_1, \cdots, b_n)$ is called a *generalized higher derivation of* *length* n from A to B if for any $t \in I$ and x, $y \in A$, the following relation is satisfied:

$$f_t(xy) = \sum_{i=0}^{t} f_i(x)b_0 f_{t-i}(y) + \sum_{i=0}^{t-1} f_i(x)b_1 f_{t-1-i}(y) + \sum_{i=0}^{t-2} f_i(x)b_2 f_{t-2-i}(y) + \cdots$$
$$+ \cdots + \sum_{i=0}^{1} f_i(x)b_{t-1} f_{1-i}(y) + f_0(x)b_t f_0(y). \tag{2.3}$$

If $b_0 = 1 \in B$, then f_0 is a ring homomorphism and if $A = B$ and $f_0 = \iota_A$, then $(f_1 \ ; \ b_1)$ is a generalized derivation in the sense of (1.7). Moreover, if $f_0 = \iota_A$, then by the relation (2.3), we have $b_0 = 1$, $b_1 = -f_1(1)$, $b_2 = f_1(1)^2 - f_2(1)$ and so on. Therefore we can not use arbitrary elements in B for generalized higher derivations $f = (f_0, f_1, \cdots, f_n \ ; \ b_0, b_1, \cdots, b_n)$. We denote that generalized higher derivation by $f = (f_t \ ; \ b_t)$. Two generalized higher derivations $f = (f_t \ ; \ b_t)$ and $g = (g_t \ ; \ c_t)$ are equal if $f_t = g_t$ and $b_t = c_t$ for any $t \in I$, and the set of all generalized higher derivations of length n from A to B is denoted by $g\mathrm{Der}^n(A, B)$. Then the corresponding results to Lemma 1.1 (1a) and (1b) are as follows.

Lemma 2.2. (1) *If* $f = (f_t \ ; \ b_t) \in g\mathrm{Der}^n(A, B)$, *then there exists a higher derivation* $d = (d_t) \in \mathrm{Der}^n(A, B)$ *such that* $f = (f_t) \in d\text{-}\mathrm{Der}^n(A, B)$.

(2) *If* $d = (d_t) \in \mathrm{Der}^n(A, B)$, *then there exists* $f = (f_t) \in d\text{-}\mathrm{Der}^n(A, B)$.

Proof. (1) Take $d_0 = (b_0)_\ell f_0$, and for any t $(1 \leq t \leq n)$, we set

$$d_t = (b_0)_\ell f_t + (b_1)_\ell f_{t-1} + (b_2)_\ell f_{t-2} + \cdots + (b_{t-1})_\ell f_1 + (b_t)_\ell f_0. \tag{2.4}$$

Then $d = (d_t) \in \mathrm{Der}^n(A, B)$, and $f = (f_t)$ and $d = (d_t)$ satisfy the relation (2.2).

(2) For a higher derivation $d = (d_t) \in \mathrm{Der}^n(A, B)$ and b_0, $b_1, \cdots$, $b_n \in B$, we define $f_0 = (b_0)_\ell d_0$ and for any $t \in I$,

$$f_t = (b_0)_\ell d_t + (b_1)_\ell d_{t-1} + \cdots + (b_t)_\ell d_0. \tag{2.5}$$

Then $f = (f_t)$ is contained in $d\text{-}\mathrm{Der}^n(A, B)$.

Since a derivation is a Jordan derivation, a higher Jordan derivation $J = (J_t)$, a higher J-Jordan derivation and a generalized higher Jordan derivation are similarly defined.

Definition 2.3. (1) Let $J_t : A \to B$ be k-linear maps ($t \in I$). $J = (J_t)$ is called a *higher Jordan derivation of length* n from A to B if for any $t \in I$ and $x \in A$, there holds

$$J_t(x^2) = \sum_{i=0}^{t} J_i(x)J_{t-i}(x). \tag{2.6}$$

(2) Let M be a right A/k-module, N a right B/k-module, $J = (J_t) \in \mathrm{JDer}^n(A, B)$ and $f_t : M \to N$ k-linear maps ($t \in I$). $f = (f_t)$ is called a *higher* J*-Jordan derivaiton of length* n from M to N if for any $t \in I$ and $x \in A$, $\omega \in M$, there holds

$$f_t(\omega x) = \sum_{i=0}^{t} f_i(\omega)J_{t-i}(x). \tag{2.7}$$

(3) For k-linear maps $f_t : A \to B$ and elements $b_0, b_1, b_2, \cdots, b_t \in B$, $f = (f_t\ ;\ b_t)$ is called a *generalized higher Jordan derivation of length* n if for any $t \in I$ and $x \in A$, there holds

$$\begin{aligned} f_t(x^2) = {} & \sum_{i=0}^{t} f_i(x)b_0f_{t-i}(x) + \sum_{i=0}^{t-1} f_i(x)b_1f_{t-1-i}(x) + \sum_{i=0}^{t-2} f_i(x)b_2f_{t-2-i}(x) + \cdots \\ & + \sum_{i=0}^{1} f_i(x)b_{t-1}f_{1-i}(x) + f_0(x)b_tf_0(x). \end{aligned} \tag{2.8}$$

If $b_0 = 1 \in B$, then J_0 is a Jordan homomorphism, i.e., $J_0(x^2) = J_0(x)J_0(x)$, and if $A = B$ and $J_0 = \iota_A$, then we have the similar results for generalized higher Jordan derivations to generalized higher derivations. We denote $\mathrm{JDer}^n(A, B)$ (resp. $g\mathrm{JDer}^n(A, B)$) the set of all higher Jordan (resp. generalized higher Jordan) derivations of length n from A to B, respectively. The set of all higher J-Jordan derivations J-$\mathrm{JDer}^n(M, N)$ is similarly defined. By using (2.4) and (2.5), we can get the following result which corresponds to Lemma 2.2.

Lemma 2.4. (1) *If* $f = (f_t\ ;\ b_t) \in g\mathrm{JDer}^n(A, B)$, *then there exists* $J = (J_t) \in \mathrm{JDer}^n(A, B)$ *such that* $f = (f_t) \in J$-$\mathrm{JDer}^n(A, B)$.

(2) *If* $J = (J_t) \in \mathrm{JDer}^n(A, B)$, *then there exists* $f = (f_t) \in J$-$\mathrm{JDer}^n(A, B)$.

Finally, we treat the case of Lie derivations.

Definition 2.5. (1) Let B be an A/k-bimodule and $L_t : A \to B$. $L = (L_t)$ is called a *higher Lie derivation* if for any $t \in I$ and x, $y \in A$, there holds

$$L_t([x,y]) = \sum_{i=0}^{t} [L_i(x), L_{t-i}(y)]. \tag{2.9}$$

We denote $\mathrm{LDer}^n(A,B)$ the set of all higher Lie derivations of length n from A to B.

(2) Let M be an A/k-bimodule, N a B/k-bimodule, $L = (L_t) \in \mathrm{LDer}^n(A,B)$ and $f_t : M \to N$ k-linear maps ($t \in I$). $f = (f_t)$ is called a *higher L-Lie* derivation of length n from M to N if for any $t \in I$ and $x \in A$, $\omega \in M$, there holds,

$$f_t([\omega,x]) = \sum_{i=0}^{t} [f_i(\omega), L_{t-i}(x)]. \tag{2.10}$$

(3) For k-linear maps $f_t : A \to B$ and elements b_0, b_1, b_2, $\cdots$, $b_t \in B$, $f = (f_t\ ;\ b_t)$ is called a *generalized higher Lie derivation* if for any $t \in I$ and x, $y \in A$, there holds

$$\begin{aligned} f_t([x,y]) = {} & \sum_{i=0}^{t} [f_i(x), b_0, f_{t-i}(y)] + \sum_{i=0}^{t-1} [f_i(x), b_1, f_{t-1-i}(y)] \\ & + \sum_{i=0}^{t-2} [f_i(x), b_2, f_{t-2-i}(y)] + \cdots + \sum_{i=0}^{1} [f_i(x), b_{t-1}, f_{1-i}(y)] \\ & + [f_0(x), b_t, f_0(y)]. \end{aligned} \tag{2.11}$$

If $b_0 = 1 \in B$, then L_0 is a Lie homomorphism, that is, $L_0([x,y]) = [L_0(x), L_0(y)]$, and if $A = B$ and L_0 is the identity map, then we also have the similar results for generalized higher Lie derivations to generalized higher derivations. But for the information of the elements b_1, $b_2, \cdots, b_t$, we only know that $f_1(1) + b_1$ is contained in the center of B. We denote L-$\mathrm{LDer}^n(M,N)$ (resp. $g\mathrm{LDer}^n(A,B)$) the set of all higher L-Lie (resp. generalized higher Lie) derivations of length n from M to N (resp. A to B).

Lemma 2.6. (1) *If $f = (f_t\ ;\ b_t) \in g\mathrm{LDer}^n(A,B)$, then there exists a higher Lie derivation $L = (L_t) \in \mathrm{LDer}^n(A,B)$. Moreover, if b_t is contained in the center of B for any $t \in I$, then $f = (f_t) \in L$-$\mathrm{LDer}^n(A,B)$.*

(2) *Let $L = (L_t)$ be in $LDer^n(A,B)$. If the center of B is non-zero, then there exists $f = (f_i) \in L$-$\mathrm{LDer}^n(A,B)$.*

Proof. In (1), we may define $L = (L_t)$ by (2.4) and for (2), we can take elements b_1 in the center of B.

If we consider these higher derivations of length 1, then the definitons in §1 from (1.1) to (1.9) are generalized as follows:

$$d_1(xy) = d_1(x)d_0(y) + d_0(x)d_1(y) \tag{1.1.1}$$

$$J_1(x^2) = J_1(x)J_0(x) + J_0(x)J_1(x) \tag{1.2.1}$$

$$L_1([x, y]) = [L_1(x), L_0(y)] + [L_0(x), L_1(y)] \tag{1.3.1}$$

$$f_1(\omega x) = f_1(\omega)d_0(x) + f_0(\omega)d_1(x) \tag{1.4.1}$$

$$g_1(x^2) = g_1(x)J_0(x) + g_0(x)J_1(x) \tag{1.5.1}$$

$$h_1([x, y]) = [h_1(x), L_0(y)] + [h_0(x), L_1(y)]. \tag{1.6.1}$$

$$f_1(xy) = f_1(x)b_0f_0(y) + f_0(x)b_0f_1(y) + f_0(x)b_1f_0(y) \tag{1.7.1}$$

$$g_1(x^2) = g_1(x)b_0J_0(x) + g_0(x)b_0J_1(x) + g_0(x)b_1g_0(x) \tag{1.8.1}$$

$$h_1([x, y]) = [h_1(x), b_0, L_0(y)] + [h_0(x), b_0, L_1(y)] + [h_0(x), b_1, h_0(y)]. \tag{1.9.1}$$

We will be treat these deformed derivations anywhere.

3. Generalized higher derivations of length 2

Now, we discuss generalized higher derivations of length 2. In general, it is complicated, and so we treat the special case which is a natural extension of generalized derivations in §1. Let A has an identity element 1 and set $\mathrm{Der}^2(A) = \mathrm{Der}^2(A, A)$ (resp. $\mathrm{gDer}^2(A) = \mathrm{gDer}^2(A, A)$). Define

$$G(\mathrm{Der}^2(A)) = \{d = (\iota_A, d_1, d_2) \in \mathrm{Der}^2(A)\}$$

$$G(g\mathrm{Der}^2(A)) = \{f = (\iota_A, f_1, f_2\ ;\ 1, a_1, a_2) \in g\mathrm{Der}^2(A)\}.$$

Define a multiplication $\star$ in $G(g\mathrm{Der}^2(A))$ as follows. For $f = (f_i\ ;\ a_i)$, $g = (g_i\ ;\ b_i) \in G(g\mathrm{Der}^2(A))$,

$$f \star g = (\iota_A, f_1 + g_1, f_2 + f_1g_1 + g_2\ ;\ 1, a_1 + b_1, a_1b_1 + b_1a_1 + a_2 + b_2 + f_1(b_1)). \tag{3.1}$$

Then we can see that $f \star g$ is contained in $G(g\mathrm{Der}^2(A))$ and $(f \star g) \star h = f \star (g \star h)$. Moreover,

$$(\iota_A, f_1, f_2\ ;\ 1, a_1, a_2)^{-1} = (\iota_A, -f_1, f_1^2 - f_2\ ;\ 1, -a_1, 2a_1^2 - a_2 + f_1(a_1))$$

is contained in $G(g\mathrm{Der}^2(A))$. Thus $G(g\mathrm{Der}^2(A))$ is a (non-commutative) group with identity $(\iota_A, 0, 0\ ;\ 1, 0, 0)$. Since a higher derivation $d = (\iota_A, d_1, d_2)$ is identified to the generalized higher derivations $(\iota_A, d_1, d_2\ ;\ 1, 0, 0)$ in $G(g\mathrm{Der}^2(A))$, $G(\mathrm{Der}^2(A))$ is considered as a subgroup of $G(g\mathrm{Der}^2(A))$. Next, we define a group structure $\circ$ on $A \oplus A$ as follows: For any $A \oplus A \ni (a_1, a_2), (b_1, b_2)$,

$$(a_1, a_2) \circ (b_1, b_2) = (a_1 + b_1, a_2 + b_2 + b_1 a_1).$$

Then $A \oplus A$ is a group with identity $(0, 0)$ and $(a_1, a_2)^{-1} = (-a_1, a_1^2 - a_2)$.

Under these notations , we have the following which corresponds to the exact sequence (1.10) of generalized derivations.

Theorem 3.1. *The follwong sequence is split exact as groups* :

$$0 \longrightarrow A \oplus A \xrightarrow{\psi_A} G(g\mathrm{Der}^2(A)) \xrightarrow{\varphi_A} G(\mathrm{Der}^2(A)) \longrightarrow 1,$$

where

$$\begin{aligned} \psi_A(a_1, a_2) &= (\iota_A, (-a_1)_\ell, (a_1^2 - a_2)_\ell\ ;\ 1, a_1, a_2), \\ \varphi_A(\iota_A, f_1, f_2\ ;\ 1, a_1, a_2) &= (\iota_A, f_1 + (a_1)_\ell, f_2 + (a_1)_\ell f_1 + (a_2)_\ell). \end{aligned}$$

This means that $G(\mathrm{Der}^2(A))$ *acts on* $A \oplus A$ *by* $(\iota_A, d_1, d_2) \rightharpoonup (a_1, a_2) = (a_1, a_2 + d_1(a_1))$ *and* $G(\mathrm{Der}^2(A))$ *is a semidirect product of* $A \oplus A$ *and* $G(\mathrm{Der}^2(A))$.

Proof. It is easy to see that ψ_A is well defined group monomorphism. And by Lemma 2.2, φ_A is also well defined and $\varphi_A \psi_A = 0$. Let $d = (\iota_A, d_1, d_2)$ be in $G(\mathrm{Der}^2(A))$ and $a_1,\ a_2 \in A$. We set $f_1 = d_1 + (a_1)_\ell$ and $f_2 = d_2 + (a_1)_\ell d_1 + (a_2)_\ell$. Then we can see that $f = (\iota_A, f_1, f_2\ ;\ 1, -a_1, -a_2 + a_1^2)$ is a generalized higher derivation of length 2 and $\varphi_A(f) = d$. Thus φ_A is an epimorphism. Moreover, for $g = (g_i\ ;\ s_i)$, $h = (h_i\ ;\ t_i) \in G(g\mathrm{Der}^2(A))$, we see that the first and the second components of $\varphi_A(g) \star \varphi_A(h)$ and $\varphi_A(g \star h)$ is equal. And the third componets of them are

$$g_2 + (s_1)_\ell g_1 + (s_2)_\ell + (g_1 + (s_1)_\ell)(h_1 + (t_1)_\ell) + h_2 + (t_1)_\ell h_1 + (t_2)_\ell$$

and

$$g_2 + g_1 h_1 + h_2 + (s_1 + t_1)_\ell (g_1 + h_1) + (s_1 t_1 + t_1 s_1 + s_2 + t_2 + g_1(t_1))_\ell,$$

respectively. Therefore $\varphi_A(g) \star \varphi_A(h) = \varphi_A(g \star h)$ if and only if $g_1(t_1)_\ell = (t_1)_\ell g_1 + (t_1 s_1)_\ell + (g_1(t_1))_\ell$. But since $(g_1\ ;\ s_1)$ is a generalized derivation, we have $g_1(t_1 x) =$

$g_1(t_1)x+t_1g_1(x)+t_1s_1x$, which shows that φ_A is a group homomorphism. Ker$\varphi_A \subset$ Imξ_A and the splitness of the sequence are clear.

For generalized higher Jordan derivations of length 2, we also define $G(\mathrm{JDer}^2(A))$ and $G(g\mathrm{JDer}^2(A))$ similarly. Then we have

Theorem 3.2. *$G(g\mathrm{JDer}^2(A))$ is a group by the multiplication $\star$ and $G(\mathrm{JDer}^2(A))$ is a subgroup of $G(g\mathrm{JDer}^2(A))$. Moreover the following sequence is split exact as groups :*

$$0 \longrightarrow A \oplus A \xrightarrow{\psi_A} G(g\mathrm{JDer}^2(A)) \xrightarrow{\varphi_A} G(\mathrm{JDer}^2(A)) \longrightarrow 1,$$

where ψ_A and φ_A are defined in Theorem 3.1.

Proof. Since generalized derivations are generalized Jordan derivations, and by Lemma 2.4, it is enough to show that $G(g\mathrm{JDer}^2(A))$ is closed by multiplication $\star$. Let $f = (f_i\ ;\ a_i)$, $g = (g_i\ ;\ b_i)$ be in $G(g\mathrm{JDer}^2(A))$. Since the multiplication is given by

$$f \star g = (\iota_A, f_1 + g_1, f_2 + f_1g_1 + g_2\ ;\ 1, a_1 + b_1, a_1b_1 + b_1a_1 + a_2 + b_2 + f_1(b_1)),$$

we have to show that $(f_2 + f_1g_1 + g_2)(x^2)$ is well computed. For any x, $y \in A$, computing $g_1((x + y)^2)$, we have

$$g_1(xy + yx) = g_1(x)y + xg_1(y) + g_1(y)x + yg_1(x).$$

Since $g_1(x^2) = g_1(x)x + xg_1(x)$, we can compute $(f_2 + f_1g_1 + g_2)(x^2)$ like a derivation. Thus $(G(\mathrm{JDer}^2(A)), \star)$ is a group.

It is not known that $G(g\mathrm{LDer}^2(A)) = \{(f_i\ ;\ a_i) \in g\mathrm{LDer}^2(A) \mid f_0 = \iota_A, a_0 = 1\}$ is a group or not by the multiplication $\star$ and so we can not get the similar result of an exact sequence of groups for generalized Lie derivations of length 2.

Remark 3.3. In general, we can define a multiplication on $g\mathrm{Der}^1(A)$ and $g\mathrm{Der}^2(A)$. For example, we consider the case of $g\mathrm{Der}^1(A)$. Let $f = (f_0, f_1\ ;\ a_0, a_1)$ and $g = (g_0, g_1\ ;\ b_0, b_1)$ be in $g\mathrm{Der}^1(A)$. Define a multiplication $\diamond$ by

$$f \diamond g = (f_0g_0, f_1g_0 + f_0g_1\ ;\ a_0f_0(b_0)a_0, a_1f_0(b_0)a_0 + a_0f_0(b_1)a_0 + a_0f_1(b_0)a_0 + a_0f_0(b_0)a_1).$$

Then we can see that $f \diamond g$ is contained in $g\mathrm{Der}^1(A)$ and $g\mathrm{Der}^1(A)$ is a semigroup. If A has an identity and $f_0 = g_0 = \iota_A$, then by definition of generalized higher derivation, $a_0 = b_0 = 1$ and f_1 is a generalized derivation. So the above multiplication $\diamond$ is given by

$$(\iota_A, f_1\ ;\ 1, a_1) \diamond (\iota_A, g_0\ ;\ 1, b_1) = (\iota_A, f_1 + g_1\ ;\ 1, a_1 + b_1 + f_1(1) + a_1).$$

Since f_1 is a generalized derivation, we have $f_1(1)+a_1 = 0$. Therefore the multiplication $\diamond$ equals $\star$ and $G(g\mathrm{Der}^1(A))$ is a subgroup of the semigroup $g\mathrm{Der}^1(A)$ by the mutiplication $\diamond$.

4. Higher Jordan derivations

In this section, we generalize some results of higher derivations by Ribenboim [R] to higher Jordan derivations.

Theorem 4.1. *Let $J = (J_t)$ and $K = (K_t)$ be in* $\mathrm{JDer}^n(A)$. *Define a multiplication $\star$ by*

$$J \star K = (L_t), \quad \text{where} \quad L_t = \sum_{i+j=t} J_i K_j.$$

Then $\mathrm{JDer}^n(A)$ *is a semigroup with identity* $\mathbf{1} = (\iota_A, 0, \cdots, 0)$.

Proof. To show that $J \star K = (L_t)$ is a higher Jordan derivation of length n, we first prove that

$$J_t(xy + yx) = \sum_{i=0}^{t} (J_i(x)J_{t-i}(y) + J_i(y)J_{t-i}(x)) \tag{4.1}$$

by induction. If $t = 0$, then J_0 is a Jordan homomorphism and so $J_0(xy + yx) = J_0(x)J_0(y) + J_0(y)J_0(x)$ is already known. Assume that (4.1) holds for less than t. Then by definition, we have $J_t(x^2) = \sum_{i=0}^{t} J_i(x)J_{t-i}(x)$ and by

$$\begin{aligned} J_t((x+y)^2) &= J_t(x^2) + J_t(y^2) + J_t(xy + yx) \\ &= \sum_{i=0}^{t} J_i(x+y)J_{t-i}(x+y) \\ &= \sum_{i=0}^{t} (J_i(x)J_{t-i}(x) + J_i(x)J_{t-i}(y) + J_i(y)J_{t-i}(x) + J_i(y)J_{t-i}(y)), \end{aligned}$$

(4.1) also holds. Now, we calculate $L_t(x^2)$. Noticing the definition of $J_t(x^2)$ is symmetric with respect to J_i, we have

$$\begin{aligned} L_t(x^2) &= \sum_{i=0}^{t} J_i K_{t-i}(x^2) = \sum_{i=0}^{t} J_i(\sum_{r=0}^{t-i} K_r(x) K_{t-i-r}(x)) \\ &= \sum_{i=0}^{t} J_i(K_0(x)K_{t-i}(x) + K_{t-i}(x)K_0(x) + K_1(x)K_{t-i-1}(x) \\ &+ K_{t-i-1}(x)K_1(x) + \cdots). \end{aligned}$$

Then by (4.1), we get

$$J_i(K_r(x)K_s(x)+K_s(x)K_r(x)) = \sum_{m=0}^{i}(J_m K_r(x) J_{i-m} K_s(x) + J_m K_s(x) J_{i-m} K_r(x)) \quad (4.2)$$

and

$$J_i(K_t(x)K_t(x)) = \sum_{m=0}^{i}(J_m K_t(x) J_{i-m} K_t(x) + J_m K_t(x) J_{i-m} K_t(x)). \quad (4.3)$$

Substituting (4.2) and (4.3) to the above $L_t(x^2)$ and rearranging, we have

$$L_t(x^2) = L_0(x)L_t(x) + L_1(x)L_{t-1}(x) + \cdots + L_t(x)L_0(x).$$

Thus $J \star K = (L_t)$ is a higher Jordan derivation of length n. Associativity of the multiplication is proved by the similar way, and $\mathbf{1} = (\iota_A, 0, \cdots, 0)$ is an identity element.

Lemma 4.2. *Let $J = (J_t) \in \mathrm{JDer}^n(A)$. Then there exists $K = (K_t) \in \mathrm{JDer}^n(A)$ such that $J \star K = K \star J = \mathbf{1}$ if and only if J_0 is bijective.*

Proof. It is enough to prove that the only if part. Let J_0 be bijective. We set $K_0 = J_0^{-1}$ and define inductively

$$\begin{aligned} K_1 &= -J_0^{-1} J_1 K_0 = -J_0^{-1} J_1 J_0^{-1}, \\ K_2 &= -J_0^{-1}(J_2 K_0 + J_1 K_1) = -J_0^{-1}(J_2 J_0^{-1} - J_1 J_0^{-1} J_1 J_0^{-1}), \\ &\cdots \\ K_\ell &= -J_0^{-1}(J_\ell K_0 + J_{\ell-1} K_1 + \cdots + J_1 K_{\ell-1}) \\ &\cdots . \end{aligned}$$

Then it is easy to see that K_0 is a Jordan homomorphism and

$$\begin{aligned}K_1(x^2) &= -J_0^{-1}J_1J_0^{-1}(x^2) = -J_0^{-1}(J_1(J_0^{-1}(x))x + xJ_1(J_0^{-1}(x)))\\ &= K_1(x)K_0(x) + K_0(x)K_1(x).\end{aligned}$$

So by the similar computation to the proof of Lemma 4.1, we see that $K = (K_t) \in \mathrm{JDer}^n(A)$ and $J \star K = K \star J = \mathbf{1}$.

The above higher Jordan derivation $J = (J_t)$ of length n is called *invertible* and we set

$$\begin{aligned}G(\mathrm{JDer}^n(A)) &= \{J = (J_t) \in \mathrm{JDer}^n(A) \mid J \text{ is invertible}\}\\ G_0(\mathrm{JDer}^n(A) &= \{J = (J_t) \in \mathrm{JDer}^n(A) \mid J_0 = \iota_A\}.\end{aligned}$$

Then $G(\mathrm{JDer}^n(A))$ is a group and $G_0(\mathrm{JDer}^n(A))$ is a subgroup of $G(\mathrm{JDer}^n(A))$.

A bijective Jordan homomorphism $J : A \to A$ is called a Jordan *automorphism*. We denote $\mathrm{JAut}(A)$ the set of all Jordan automorphism of A. Under these notations, the following theorem is easily proved.

Theorem 4.3. *The sequence of groups*

$$1 \longrightarrow G_0(\mathrm{JDer}^n(A)) \xrightarrow{i} G(\mathrm{JDer}^n(A)) \xrightarrow{\varphi} \mathrm{JAut}(A) \longrightarrow 1 \tag{4.4}$$

is split exact, where i is the canonical injection and $\varphi(J = (J_t)) = J_0$.

We can generalize the above results to the set of all higher J-Jordan derivations as follows.

Lemma 4.4. *Let M be a right A/k-module, $f = (f_t) \in J$-$\mathrm{JDer}^n(M)$ and $g = (g_t) \in K$-$\mathrm{JDer}^n(M)$. Then $f \star g = (h_t)$ is a higher $J \star K$-Jordan derivation of length n, where $h_t = (f \star g)_t = \sum_{i+k=\ell} f_i g_k$.*

Proof. By the following computation

$$\begin{aligned}
(f \star g)_t(\omega x) &= \sum_{i=0}^{t}\sum_{r=0}^{t-i} f_i(g_r(\omega)K_{t-i-r}(x)) \\
&= \sum_{i=0}^{t}\sum_{j=0}^{i}\sum_{r=0}^{t-i} f_j g_r(\omega) J_{i-j} K_{t-i-r}(x) \\
&= \sum_{r=0}^{t} f_0 g_r(\omega) J_0 K_{t-r}(x) + \sum_{r=0}^{t-1}\sum_{j=0}^{1} f_j g_r(\omega) J_{1-j} K_{t-1-r}(x) + \cdots \\
&+ \sum_{r=0}^{t-\ell}\sum_{j=0}^{\ell} f_j g_r(\omega) J_{\ell-j} K_{t-\ell-r}(x) + \cdots + \sum_{j=0}^{t} f_j g_0(\omega) J_{t-j} K_0(x) \\
&= \sum_{i=0}^{t} f_0 g_0 J_i K_{t-i} + \sum_{i=0}^{t-1}\sum_{j=0}^{1} f_j(g_{1-j}(\omega) J_i K_{t-1-i}(x)) + \cdots \\
&+ \sum_{i=0}^{t-\ell}\sum_{j=0}^{\ell} f_j g_{\ell-j}(\omega) J_i K_{t-\ell-i}(x) + \cdots + \sum_{j=0}^{t} f_j g_{t-j}(\omega) J_0 K_0(x) \\
&= \sum_{i=0}^{t}\sum_{s=0}^{t-i}\sum_{j=0}^{i} f_j g_{i-j}(\omega) J_s K_{t-i-s}(x) = \sum_{i=0}^{t} (f \star g)_i (J \star K)_{t-i}(\omega x),
\end{aligned}$$

$f \star g$ is a higher $J \star K$-Jordan derivation.

Now let

$$\mathbb{J}\mathrm{Der}^n(M) = \cup J\text{-}\mathrm{JDer}^n(M)$$

the set of unions of all J-$\mathrm{JDer}^n(M)$, where $J \in \mathrm{JDer}^n(A)$. Then by Theorem 4.1 and Lemma 4.4, the multiplication $\star$ in $\mathbb{J}\mathrm{Der}^n(M)$ is well-defined, and thus $\mathbb{J}\mathrm{Der}^n(M)$ is a semigroup with identity $\mathbf{1} = (\iota_M, 0, \cdots, 0)$, where ι_M is the identity map on M. Then there holds the following result which corresponds to Lemma 4.2.

Lemma 4.5. *Let A/k be an algebra and M a faithful right A/k-module. Let $J = (J_t) \in \mathrm{JDer}^n(A)$ and $f = (f_t) \in J$-$\mathrm{JDer}^n(M)$. Then there exists a higher Jordan derivation $K = (K_t) \in \mathrm{JDer}^n(A)$ and a higher K-Jordan derivation $g = (g_t) \in K$-$\mathrm{JDer}^n(M)$ such that $f \star g = g \star f = \mathbf{1}$ if and only if J_0 and f_0 are bijective.*

The above higher Jordan derivation $f = (f_t)$ of length n is called *invertible* and we set

$$G(\mathbb{J}\mathrm{Der}^n(M)) = \{f = (f_t) \in \mathbb{J}\mathrm{Der}^n(M) \mid f \text{ is invertible}\}$$
$$G_0(\mathbb{J}\mathrm{Der}^n(M) = \{f = (f_t) \in \mathrm{JDer}^n(M) \mid f_0 = \iota_M\}.$$

Then $G(\mathbb{J}\mathrm{Der}^n(M))$ is a group and $G_0(\mathbb{J}\mathrm{Der}^n(M))$ is a subgroup of $G(\mathbb{J}\mathrm{Der}^n(M))$.

Let $\gamma : M \to M$ be a bijective k-linear map such that $\gamma(\omega x) = \gamma(\omega)J(x)$ for some Jordan homomorphism $J : A \to A$ ($\omega \in M$, $x \in A$). Then γ is called a *Jordan semi-automorphism.* And J-semi-Auto(M) denote the set of all Jordan semi-automorhisms from M to M. Under these notations, we have

Theorem 4.6. *Let M be a faithful right A/k-module. Then the sequence of groups*

$$1 \longrightarrow G_0(\mathbb{J}\mathrm{Der}^n(M)) \xrightarrow{i} G(\mathbb{J}\mathrm{Der}^n(M)) \xrightarrow{\varphi} \text{J-semi-Auto}(M) \longrightarrow 1 \qquad (4.5)$$

is split exact, where i is the canonical injection and $\varphi(f = (f_t)) = f_0$.

Proof. If γ is a Jordan semi-automorphism, then $\tilde{\gamma} = (\gamma, 0, \cdots, 0)$ is a J-Jordan derivation of length n and so we have the map ψ : J-semi-Auto$(M) \ni \gamma \mapsto \tilde{\gamma} \in G(\mathrm{JDer}^n(M))$ such that $\psi\varphi$ is identity on J-semi-Auto(M).

5. Higher Lie derivations

For higher Lie derivations, we do not meet with the problem for calculation like the proof of Lemma 4.1. So we only denote the corresponding results in §4 for higher Lie derivations.

Lemma 5.1. *Let $L = (L_t)$ and $K = (K_t)$ be in* $\mathrm{LDer}^n(A)$*. Define a multiplication by*

$$L \star K = (T_t), \quad \text{where} \quad T_t = \sum_{i+j=t} L_i K_j.$$

Then $\mathrm{LDer}^n(A)$ *is a semigroup with* **1**.

Lemma 5.2. *Let $L = (L_t) \in \mathrm{LDer}^n(A)$. Then there exists $K = (K_t) \in \mathrm{LDer}^n(A)$ such that $L \star K = K \star L = \mathbf{1}$ if and only if L_0 is bijective.*

The above higher Lie derivation $L = (L_t)$ of length n is called *invertible* and we denote $G(\mathrm{LDer}^n(A)) = \{L = (L_t) \in \mathrm{LDer}^n(A)\}$ the set of all invertible higher Lie derivations and $G_0(\mathrm{LDer}^n(A) = \{L = (L_t) \in \mathrm{LDer}^n(A) \mid L_0 = \iota_A\}$. Then $G(\mathrm{LDer}^n(A))$ is a group and $G_0(\mathrm{LDer}^n(A))$ is a subgroup of $G(\mathrm{LDer}^n(A))$.

A bijective Lie homomorphism $L : A \to A$ is called a Lie *automorphism*. We denote $\mathrm{LAut}(A)$ the set of all Lie automorphism of A. Then we have

Theorem 5.3. *The sequence of groups*

$$1 \longrightarrow G_0(\mathrm{LDer}^n(A)) \xrightarrow{i} G(\mathrm{LDer}^n(A)) \xrightarrow{\varphi} \mathrm{LAut}(A) \longrightarrow 1 \tag{5.1}$$

is split exact, where i is the canonical injection and $\varphi(L = (L_t)) = L_0$.

Lemma 5.4. *Let M be an A/k-bimodule, $f = (f_t) \in L$-$\mathrm{LDer}^n(M)$ and $g = (g_t) \in K$-$\mathrm{LDer}^n(M)$. Then $f \star g = (h_t)$ is a higher $L \star K$-Lie derivation of length n, where $h_t = \sum_{i+k=\ell} f_i g_k$.*

Lemma 5.5. *Let A/k be an algebra and M a faithful A/k-bimodule. Let $L = (L_t) \in \mathrm{LDer}^n(A)$ and $f = (f_t) \in L$-$\mathrm{LDer}^n(M)$. Then there exists a higher Lie derivation $K = (K_t) \in\mathrm{LDer}^n(A)$ and a higher K-Lie derivation $g \in K$-$\mathrm{LDer}^n(M)$ such that $f \star g = g \star f = \mathbf{1}$ if and only if L_0 and f_0 are bijective.*

The above higher Lie derivation $f = (f_t)$ of length n is called *invertible* and we denote $G(\mathbb{L}\mathrm{Der}^n(M))$ the set of all invertible higher L-Lie derivations for some $L \in \mathrm{LDer}^n(M)$ and $G_0(\mathbb{L}\mathrm{Der}^n(M)) = \{f = (f_i) \in \mathbb{L}\mathrm{Der}^n(M) \mid f_0 = \iota_M\}$. Then $G(\mathbb{L}\mathrm{Der}^n(M))$ is a group and $G_0(\mathbb{L}\mathrm{Der}^n(M))$ is a subgroup of $G(\mathbb{L}\mathrm{Der}^n(M))$.

Let $\gamma : M \to M$ be a bijective k-linear map such that $\gamma(\omega x) = \gamma(\omega)L(x)$ for some Lie homomorphism $L : A \to A$. Then γ is called a *Lie semi-automorphism.* And L-semi-Auto(M) denote the set of all Lie semi-automorhisms from M to M. Under these notations, we finally get the following

Theorem 5.6. *Let M be a faithful A/k-bimodule. Then the sequence of groups*

$$1 \longrightarrow G(\mathbb{L}\mathrm{Der}^n(M)) \xrightarrow{i} G(\mathbb{L}\mathrm{Der}^n(M)) \xrightarrow{\varphi} \text{L-semi-Auto}(M) \longrightarrow 1 \tag{5.2}$$

is split exact, where i is the canonical injection and $\varphi(f) = f_0$.

Although we define some types of generalized higher derivations, we can not treat the structure of $g\mathrm{Der}^n(A)$, $g\mathrm{JDer}^n(A)$ and $g\mathrm{LDer}^n(A)$ in our method. But we will give the similar exact sequence for the case of length 2.

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