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Oscillation Criteria for Second Order Nonlinear Differential Equations with Damping

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Abstract

Oscillation criteria are given for second order nonlinear differential equations with damping of the form

$$(a(t)\psi(x)\dot{x})^{\cdot} + p(t)\dot{x} + q(t)f(x) = 0, \quad t \geq t_0,$$

where p and q are allowed to change signs on $[t_0, \infty)$. We employ the averaging technique to obtain sufficient conditions for oscillation of solutions of the above equation. Our results generalize and extend some known oscillation criteria in the literature.
Key words and phrases: Oscillation, averaging, damping, Riccati substitution, second order.

1. Introduction

We are concerned with the oscillation of solutions of second order differential equations with damping terms of the following form

$$(a(t)\psi(x)\dot{x})^{\cdot} + p(t)\dot{x} + q(t)f(x) = 0, \quad t \geq t_0, \tag{1.1}$$

where $a \in C[[t_0, \infty), R_+]$, $p, q \in C[[t_0, \infty), R]$, $\psi \in C[R, R_+]$, and $f \in C^1[R, R]$. We shall assume that $xf(x) > 0$ for $x \neq 0$, and that for $x \in R$

$$f'(x) \geq k, \tag{1.2}$$

and

$$c \leq \psi(x) \leq c_1, \tag{1.3}$$

where k, c, and c_1 are some positive real numbers.

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As is customary, a solution $x(t)$ of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1.1) is oscillatory if all of its solutions are oscillatory. It is tacitly assumed that equation (1.1) has nontrivial solutions which exist for all $t \geq t_0$.

In the special case $a(t) \equiv 1$, $\psi(x) \equiv 1$, $p(t) \equiv 0$, and $f(x) = x$, equation (1.1) reduces to the linear second order differential equation

$$x'' + q(t)x = 0. \tag{L}$$

Equation (L) has been investigated by several authors. Below is a list of some well known oscillation criteria for equation (L) that exist in the literature:

(Leighton [5]): $\int_{t_0}^{\infty} q(t)\,dt = \infty.$

(Wintner [8]): $\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{r} q(r)\,dr\,ds = \infty.$

(Hartman [3]): $-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{r} q(r)\,dr\,ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{r} q(r)\,dr\,ds \leq \infty.$

(Kamenev [4]): $\lim_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}(t-s)^m q(s)\,ds = \infty$ for some integer $m > 1$.

(Yan [10]): $\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}(t-s)^m q(s)\,ds = \infty$, $\limsup_{t\to\infty} \int_{T}^{t}(t-s)^m q(s)\,ds > A(T)$ for all $T \geq t_0$, where $A(u)$ is a continuous function such that $\int_{t_0}^{t} A_+(u)\,du = \infty$ with $A_+(u) = \max\{A(u), 0\}$.

(Philos [7]): $H(t,s) \in C(D,R)$, $D = \{(t,s) : t \geq s \geq t_0\}$, has a continuous and nonpositive partial derivative on D with respect to the second variable and satisfies $H(t,t) = 0$ for $t \geq t_0, H(t,s) > 0$ for $t > s \geq t_0$; there exists a continuous function $h : D \to R$ such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right] ds = \infty.$$

Recently, Grace [1, 2], using the averaging technique and the arguments developed by Philos, gave some oscillation criteria concerning the solutions of (1.1).

In this paper we shall also make use of Philos's technique to establish new oscillation criteria for equation (1.1). Our theorems improve and generalize several results obtained previously.

2. Main Results

In what follows, $Q(t)$ denotes

$$Q(t) = q(t) - \frac{1}{4k}\left(\frac{1}{c} - \frac{1}{c_1}\right)\frac{p^2(t)}{a(t)}, \quad t \geq t_0.$$

Theorem 2.1 *Let conditions (1.2) and (1.3) hold, and* $D = \{(t,s) : t \geq s \geq t_0\}$. *Let* $H \in C(D, R)$ *satisfy the following two conditions:*

(i) $H(t,t) = 0$ *for* $t \geq t_0, H(t,s) > 0$ *for* $t > s \geq t_0$;

(ii) H *has a continuous and nonpositive partial derivative on* D *with respect to the second variable.*

If there exist an $h \in C(D, R)$ *and a differentiable function* $\rho : [t_0, \infty) \to (0, \infty)$ *such that*

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D \tag{2.1}$$

and

$$\limsup_{t\to\infty} [X(t,t_0) - \frac{1}{4kc_1}Y(t,t_0)] = \infty, \tag{2.2}$$

where

$$X(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\rho(s)Q(s)\,ds$$

$$Y(t,t_0) = \frac{1}{H(t,t_0)}\int_{t_0}^{t} a(s)\rho(s)\left\{\left(\frac{p(s)}{a(s)} - \frac{c_1\dot{\rho}(s)}{\rho(s)}\right)\sqrt{H(t,s)} + c_1 h(t,s)\right\}^2 ds,$$

then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$ for all $t \geq T_0$ for some $T_0 \geq t_0$. The proof when $x(t) < 0$ for $t \geq T_0$ is similar.

We define

$$W(t) = \rho(t)\frac{a(t)\psi(x(t))\dot{x}(t)}{f(x(t))}, \quad t \geq T_0. \tag{2.3}$$

Differentiating (2.3) and using (1.1) and (1.2), we see that

$$\begin{aligned}\dot{W}(t) &= -\rho(t)q(t) - \frac{p(t)}{a(t)}\frac{1}{\psi(x(t))}W(t) + \frac{\dot{\rho}(t)}{\rho(t)}W(t) - \frac{1}{a(t)\rho(t)}\frac{f'(x(t))}{\psi(x(t))}W^2(t) \\ &\leq -\rho(t)q(t) + \frac{p^2(t)\rho(t)}{4ka(t)\psi(x(t))} + \frac{\dot{\rho}(t)}{\rho(t)}W(t) \\ &\quad -\frac{1}{\psi(x(t))}\left[\sqrt{\frac{k}{a(t)\rho(t)}}W(t) + \frac{p(t)\sqrt{\rho(t)}}{2\sqrt{ka(t)}}\right]^2.\end{aligned}$$

Using (1.3) in the above inequality, it follows that

$$\dot{W}(t) \leq -\rho(t)Q(t) - \frac{1}{c_1}\left[\frac{k}{a(t)\rho(t)}W^2(t) + r(t)W(t)\right], \quad r(t) = \frac{p(t)}{a(t)} - \frac{c_1\dot{\rho}(t)}{\rho(t)},$$

and hence, in view of (i) and (ii), for $t \geq T \geq T_0$, we have

$$\begin{aligned}\int_T^t H(t,s)\rho(s)Q(s)ds &\leq H(t,T)W(T) \\ &\quad -\frac{1}{c_1}\int_T^t\left[\frac{kH(t,s)}{a(s)\rho(s)}W^2(s) + \left\{c_1h(t,s)\sqrt{H(t,s)} + r(s)H(t,s)\right\}W(s)\right]ds \\ &= H(t,T)[W(T) - J(t,T)] + \int_T^t \frac{a(s)\rho(s)}{4kc_1}\left[r(s)\sqrt{H(t,s)} + c_1h(t,s)\right]^2 ds, \end{aligned} \tag{2.4}$$

where

$$J(t,T) = \frac{1}{c_1H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{a(s)\rho(s)}}W(s) + \sqrt{\frac{a(s)\rho(s)}{4k}}\left\{r(s)\sqrt{H(t,s)} + c_1h(t,s)\right\}\right]^2 ds.$$

Moreover, (2.4) implies that for $t \geq T_0$,

$$H(t,T_0)[X(t,T_0) - \frac{1}{4kc_1}Y(t,T_0)] \leq H(t,T_0)W(T_0) \leq H(t,t_0)|W(T_0)|. \tag{2.5}$$

In view of (2.4) and (2.5), one can easily obtain that

$$
\begin{aligned}
&H(t,t_0)[X(t,t_0) - \frac{1}{4kc_1}Y(t,t_0)] = \\
&\quad \int_{t_0}^{T_0}\left[H(t,s)\rho(s)Q(s) - \frac{a(s)\rho(s)}{4c_1k}\left\{r(s)\sqrt{H(t,s)} + c_1h(t,s)\right\}^2\right]ds + \\
&\quad \int_{T_0}^{t}\left[H(t,s)\rho(s)Q(s) - \frac{a(s)\rho(s)}{4c_1k}\left\{r(s)\sqrt{H(t,s)} + c_1h(t,s)\right\}^2\right]ds \\
&\quad \le H(t,t_0)\int_{t_0}^{T_0}|\rho(s)Q(s)|ds + H(t,t_0)|W(T_0)|,
\end{aligned}
$$

for $t \ge T_0$, and so we have

$$
\lim_{t\to\infty}\sup[X(t,t_0) - \frac{1}{4kc_1}Y(t,t_0)] \le \int_{t_0}^{T_0}|\rho(s)Q(s)|ds + |W(T_0)|.
$$

Since this last inequality contradicts (2.2), the proof is complete.

Remark 2.1 In Theorem 2.1, if we take $H(t,s) = (t-s)^\alpha$, $\alpha > 1$, we recover Theorem 1 in [1, 6]. Also, if $\psi(x) = 1$ and $f(x) = x$, we obtain the Yan's oscillation criterion mentioned in the previous section.

A close look at the proof of Theorem 2.1 reveals that condition (2.2) may be replaced by the conditions that

$$
\limsup_{t\to\infty} X(t,t_0) = \infty \quad \text{and} \limsup_{t\to\infty} Y(t,t_0) < \infty. \tag{2.6}
$$

This leads to the following result.

Corollary 2.1 *Let the conditions of Theorem 2.1 be satisfied except that condition (2.2) is replaced by (2.6). Then equation (1.1) is oscillatory.*

Remark 2.2 It is easy to see that (2.6) implies (2.2) but not conversely. Furthermore, if we set $H(t,s) = (t-s)^\alpha$, $\alpha > 1$, $\rho(s) = s^\beta$, $\beta \in [0,1)$, $\psi(x) = 1$, $f(x) = x$ in Corollary 2.1, we obtain the oscillation theorem given by Yan in [9].

Example 2.1 Consider

$$(t^{-1}(2-\sin x)\dot{x})^{\cdot}+\sin t\,\dot{x}+t^2\cos t\,x(1+x^4)=0,\quad t\geq 1.$$

If we take $\rho(t)=t$ and $H(t,s)=(t-s)^2$ then we see that, since the conditions of Theorem 2.1 are satisfied, the equation is oscillatory. We should note that the oscillation criteria given in [1-10] fail to apply for this equation.

In Theorem 2.1 the condition

$$\limsup_{t\to\infty} X(t,t_0)=\infty$$

is necessary. In the remainder of this paper we do not require this condition and naturally will have some other conditions instead. The following result provides a different such oscillation criterion for equation (1.1).

Theorem 2.2 *Let conditions (1.2) and (1.3) hold and h and H be as in Theorem 2.1, and let*

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}>0. \tag{2.7}$$

Suppose that there exist a positive function $\rho\in C^1[t_0,\infty)$ and $A\in C[t_0,\infty)$ such that

$$\liminf_{t\to\infty} Y(t,t_0)<\infty, \tag{2.8}$$

$$\liminf_{t\to\infty}[X(t,T)-\frac{1}{4kc_1}Y(t,T)]\geq A(T),\quad \text{for every } T\geq t_0 \tag{2.9}$$

and

$$\int_{t_0}^{\infty}\frac{A_+^2(s)}{a(s)\rho(s)}ds=\infty, \tag{2.10}$$

where $A_+(s)=\max\{A(s),0\}$. Then equation (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, we see that (2.4) holds for all $t\geq T\geq t_0$. So, for $t>T\geq T_0$, we have

$$X(t,T)-\frac{1}{4kc_1}Y(t,T)\leq W(T)-J(t,T) \tag{2.11}$$

and hence

$$\lim_{t\to\infty} \inf[X(t,T) - \frac{1}{4kc_1}Y(t,T)] \le W(T) - \limsup_{t\to\infty} J(t,T) \tag{2.12}$$

for all $T \ge T_0$.

Making use of (2.9) in (2.12), it follows that

$$W(T) \ge A(T) \quad + \quad \limsup_{t\to\infty} J(t,T) \tag{2.13}$$

for all $T \ge T_0$. Thus, from (2.13), for all $T \ge T_0$,

$$A(T) \le W(T) \tag{2.14}$$

and

$$\limsup_{t\to\infty} J(t,T) < \infty \tag{2.15}$$

From (2.14),

$$\int_{T_0}^{\infty} \frac{W^2(s)}{a(s)\rho(s)} ds \ge \int_{T_0}^{\infty} \frac{A_+^2(s)}{a(s)\rho(s)} ds,$$

and hence by (2.10),

$$\int_{T_0}^{\infty} \frac{W^2(s)}{a(s)\rho(s)}\, ds = \infty \tag{2.16}$$

To complete the proof we show that (2.16) is not possible. For this purpose, we introduce the functions $u(t)$ and $v(t)$ defined for $t \ge T_0$ as follows;

$$u(t) = \frac{1}{c_1 H(t,T_0)} \int_{T_0}^{t} \frac{kH(t,s)}{a(s)\rho(s)} W^2(s) ds,$$

$$v(t) = \frac{1}{c_1 H(t,T_0)} \int_{T_0}^{t} \left[c_1 h(t,s)\sqrt{H(t,s)} + r(s)H(t,s)\right] W(s) ds,$$

where

$$r(t) = \frac{p(t)}{a(t)} - \frac{c_1 \dot{\rho}(t)}{\rho(t)}.$$

It follows from (2.15) that

$$\lim_{t\to\infty} \sup[u(t) + v(t)] < \infty. \tag{2.17}$$

Because of (2.7) one can find a positive constant M_1 such that

$$\inf_{s\geq t_0}\left(\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right)>M_1 \tag{2.18}$$

Let M_2 be an arbitrary positive real number. It follows from (2.16) that if $T_1>T_0$ is large enough then

$$\int_{T_0}^{t}\frac{W^2(s)}{a(s)\rho(s)}ds\geq\frac{c_1M_2}{kM_1}\quad\text{for all } t\geq T_1.$$

Then

$$\begin{aligned}
u(t) &= \frac{k}{c_1H(t,T_0)}\int_{T_0}^{t}H(t,s)d\left(\int_{T_0}^{s}\frac{W^2(\xi)}{a(\xi)\rho(\xi)}d\xi\right)\\
&\geq \frac{k}{c_1H(t,T_0)}\int_{T_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)\left\{\int_{T_0}^{s}\frac{W^2(\xi)}{a(\xi)\rho(\xi)}d\xi\right\}ds\\
&\geq \frac{M_2}{M_1}\frac{1}{H(t,T_0)}\int_{T_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)ds\\
&= \frac{M_2}{M_1}\frac{H(t,T_1)}{H(t,T_0)}\quad\text{for all } t\geq T_1.
\end{aligned}$$

Making use of (2.18), we see that there is a $T_2\geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)}\geq M_1\quad\text{for all } t\geq T_2.$$

Thus, we get

$$u(t)\geq M_2\quad\text{for all } t\geq T_2.$$

Since M_2 is arbitrary, this means that

$$\lim_{t\to\infty}u(t)=\infty \tag{2.19}$$

Next, we consider an arbitrary sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0,∞) with $\lim_{n\to\infty}t_n=\infty$. By (2.17), there is a number M such that

$$u(t_n)+v(t_n)\leq M\quad\text{for}\quad n=1,2,3,\ldots. \tag{2.20}$$

In view of (2.19) and (2.20),

$$\lim_{n\to\infty}u(t_n)=\infty. \tag{2.21}$$

and

$$\lim_{n\to\infty} v(t_n) = -\infty. \tag{2.22}$$

Now, because of (2.20) and (2.21), there exists a number N such that,

$$1 + \frac{v(t_n)}{u(t_n)} \le \frac{M}{u(t_n)} < \frac{1}{2}$$

or

$$\frac{v(t_n)}{u(t_n)} < -\frac{1}{2}$$

for every $n \ge N$. This and (2.22) give

$$\lim_{n\to\infty} \frac{v^2(t_n)}{u(t_n)} = \infty \tag{2.23}$$

On the other hand, by the Schwarz inequality, since

$$\begin{aligned}
v^2(t_n) &= \left\{ \frac{1}{c_1 H(t_n, T_0)} \int_{T_0}^{t_n} \left[c_1 h(t_n, s)\sqrt{H(t_n, s)} + r(s) H(t_n, s) \right] W(s) ds \right\}^2 \\
&= \left\{ \frac{1}{c_1 H(t_n, T_0)} \int_{T_0}^{t_n} \left[c_1 h(t_n, s) + r(s)\sqrt{H(t_n, s)} \right] \sqrt{H(t_n, s)} W(s) ds \right\}^2 \\
&\le \frac{1}{c_1 H(t_n, T_0)} \int_{T_0}^{t_n} \frac{k H(t_n, s)}{a(s)\rho(s)} W^2(s)\, ds \\
&\times \frac{1}{c_1 k H(t_n, T_0)} \int_{T_0}^{t_n} a(s)\rho(s) \left[c_1 h(t_n, s) + r(s)\sqrt{H(t_n, s)} \right]^2 ds \\
&= \frac{u(t_n)}{c_1 k H(t_n, T_0)} \int_{T_0}^{t_n} a(s)\rho(s) \left[c_1 h(t_n, s) + r(s)\sqrt{H(t_n, s)} \right]^2 ds,
\end{aligned}$$

we have

$$\frac{v^2(t_n)}{u(t_n)} \le \frac{1}{k c_1} Y(t_n, T_0) \quad \text{for any positive integer } n. \tag{2.24}$$

Clearly, inequality (2.18) guarantees that for n large enough,

$$\frac{H(t_n, T_0)}{H(t_n, t_0)} \ge M_1. \tag{2.25}$$

Then, by combining (2.24) and (2.25), we obtain

$$\frac{v^2(t_n)}{u(t_n)} \leq \frac{1}{M_1 k c_1} Y(t_n, t_0) \quad \text{for all large } n,$$

which, due to (2.23), implies that

$$\lim_{n\to\infty} Y(t_n, t_0) = \infty. \tag{2.26}$$

Clearly, since the sequence $\{t_n\}$ is arbitrary, (2.26) contradicts (2.8). The proof is therefore complete.

Example 2.2 Consider

$$(t^{-2}e^{2t}(1+e^{-|x|})\dot{x})^{\cdot} - 4t^{-2}e^{2t}\dot{x} + 3t^{-2}e^{2t}x(1+x^2) = 0, \quad t \geq 1.$$

It can be checked that the oscillation criteria given in [1-10] do not apply for this equation. By taking $\rho(t) = e^{-2t}$ and $H(t,s) = (t-s)^2$, we see that with $A(t) = t^{-1}$ the conditions of Theorem 2.2 are satisfied. Therefore, the equation is oscillatory.

Theorem 2.3 *Let conditions (1.2) and (1.3) hold. Let $H(t,s)$ and $h(t,s)$ be as in Theorem 2.1, and (2.7) holds. Suppose that there exist a positive function $\rho \in C^1[t_0\infty)$ and $A \in C[t_0,\infty)$ such that (2.10) and the following conditions hold:*

$$\limsup_{t\to\infty} Y(t,t_0) < \infty \tag{2.27}$$

and

$$\limsup_{t\to\infty}[X(t,t_0) - \frac{1}{4kc_1}Y(t,t_0)] \geq A(T) \quad \text{for every } T \geq t_0, \tag{2.28}$$

then equation (1.1) is oscillatory.

Proof. We proceed as in the proof of Theorem 2.2 and obtain (2.11). Taking the limit superior in (2.11) as $t \to \infty$, we obtain (2.12) except that liminf and limsup are now interchanged. Then, (2.13) through (2.19) are valid with exceptions that in (2.13), (2.15), and (2.17) we have liminf instead of limsup. Now, the sequence $\{t_n\}$ cannot be arbitrary; it is chosen such that $\lim_{n\to\infty}[u(t_n) + v(t_n)] = \liminf_{t\to\infty}[u(t) + v(t)]$. Continuing as in the proof of Theorem 2.2, one can easily see that (2.26) holds, and therefore (2.27) cannot be true. This completes the proof.

Theorem 2.4 *Let (1.2), (1.3), and (2.7) hold, and $H(t,s)$ and $h(t,s)$ be as in Theorem 2.1. If there exist a positive function $\rho \in C^1[t_0,\infty)$ and $A \in C[t_0,\infty)$ for which (2.9), (2.10), and*

$$\liminf_{t\to\infty} X(t,t_0) < \infty \tag{2.29}$$

are satisfied, then equation (1.1) is oscillatory.

Proof. From (2.9), we have

$$\begin{aligned} A(t_0) &\leq \liminf_{t\to\infty}[X(t,t_0) - \frac{1}{4kc_1}Y(t,t_0)] \\ &\leq \liminf_{t\to\infty} X(t,t_0) - \frac{1}{4kc_1}\limsup_{t\to\infty} Y(t,t_0) \\ &\leq \liminf_{t\to\infty} X(t,t_0) - \frac{1}{4kc_1}\liminf_{t\to\infty} Y(t,t_0) \end{aligned}$$

and therefore by (2.29),

$$\liminf_{t\to\infty} Y(t,t_0) < \infty.$$

The remainder of the proof now proceeds exactly as in that of Theorem 2.2.

Remark 2.3 If $\psi(x) = 1$, then Theorem 2.3 and Theorem 2.4 reduce to Theorem 6 and Theorem 7 in [2], respectively, and extend to equation (1.1) the results of Theorem 3 and Theorem 4 in [2] provided that (1.3) holds. Furthermore, when $H(t,s) = (t-s)^\alpha$, $\alpha > 1$, $\psi(x) = 1$, and $f(x) = x$, Theorem 2 in [10] is improved in the sense that $\limsup_{t\to\infty} X(t,t_0) < \infty$ is replaced by (2.29).

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