A Generalized Trapezoid Inequality for Functions of
Bounded Variation

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Abstract

We establish a generalization of a recent trapezoid inequality for functions of bounded variation. A number of special cases are considered. Applications are made to quadrature formulae, probability theory, special means and the estimation of the beta function.

Key Words: Trapezoid inequality, bounded variation, numerical integration, beta function.

1. Introduction

In [1], Dragomir proved the following trapezoid inequality for functions of bounded variation. Here and subsequently in the paper, if \( f \) is of bounded variation on \([a,b]\), we denote its total variation on that interval by \( W_a^b(f) \).

Theorem A. Let \( f : [a, b] \to \mathbb{R} \) be of bounded variation on \([a,b]\). Then

\[
\left| \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} (b-a) \right| \leq \frac{1}{2} (b-a) W_a^b(f).
\] (1.1)

The constant 1/2 is best possible.

We introduce the notation \( I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b \) for a division of the interval \([a, b]\), with \( h_i := x_{i+1} - x_i \) (0 \( \leq i \leq n \)) and \( \nu (h) := \max \{ h_i \mid i = 0, ..., n-1 \} \) for the norm of the division. Then we may deduce from Theorem A that

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\[ \int_{a}^{b} f(t)\, dt = T(f, I_n) + R(f, I_n), \]  

(1.2)

where

\[ T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i, \]  

(1.3)

and that the remainder term satisfies

\[ |R(f, I_n)| \leq \frac{1}{2} \nu(h) \sqrt{a} (f). \]  

(1.4)

Here the constant 1/2 is also best possible.

The main aim of this paper is to compare \( \int_{a}^{b} f(t)\, dt \) with

\[ f(a)(x - a) + f(b)(b - x), \]

where \( x \in [a, b] \) is a free parameter. The choice \( x = (a + b)/2 \) gives the trapezoid estimate

\[ \frac{f(a) + f(b)}{2} (b - a) \]

for mappings of bounded variation.

In Section 2 we derive our basic estimate, which provides an upper bound for the difference between \( \int_{a}^{b} f(t)\, dt \) and the estimate proposed above for the case when \( f \) is a function of bounded variation. We examine the important special cases when \( f \) has a continuous derivative or is Lipschitz, monotone or convex. In Section 3 these results are applied to the estimation of the error term in some quadrature formulæ and in Section 4 to some estimates in probability theory, in particular, that of the mean \( E(X) \) of a random variable \( X \). Section 5 uses particular choices of \( f \) to obtain some apparently new inequalities subsisting amongst various well-known means of a pair of positive numbers. Finally a further special choice is taken in Section 6 to address the estimation of Euler’s beta function.

For a compendious treatment of other inequalities of trapezoid type, see [2] and the references therein.
2. Some Integral Inequalities

We start with a basic integral inequality for mappings of bounded variation. For convenience we set

\[ J(x) := \int_a^b f(t) \, dt - f(a) (x-a) - f(b) (b-x). \]

**Theorem 1** Let \( f : [a, b] \to \mathbb{R} \) be a mapping of bounded variation. Then

\[ |J(x)| \leq \left[ \frac{1}{2} (b-a) + \left| x - \frac{a + b}{2} \right| \right] \sqrt{\int_a^b (f)} \quad (2.1) \]

for all \( x \in [a, b] \). The constant \( 1/2 \) is best possible.

**Proof.** By the integration by parts formula for a Riemann–Stieltjes integral, we have

\[ \int_a^b (x-t) \, df(t) = (x-t) f(t)|_a^b + \int_a^b f(t) \, dt, \]

whence we derive the identity

\[ \int_a^b f(t) \, dt = (b-x) f(b) + (x-a) f(a) + \int_a^b (x-t) \, df(t) \quad (2.2) \]

for all \( x \in [a, b] \).

If \( g, v : [a, b] \to \mathbb{R} \) are such that \( g \) is continuous and \( v \) of bounded variation on \([a, b]\),
then \( \int_a^b g(t) \, dv(t) \) exists and

\[ \left| \int_a^b g(t) \, dv(t) \right| \leq \sup_{t \in [a,b]} |g(t)| \sqrt{\int_a^b (v)} . \]

Thus

\[ \left| \int_a^b (x-t) \, df(t) \right| \leq \sup_{t \in [a,b]} |x-t| \sqrt{\int_a^b (f)} . \quad (2.3) \]
As

\[
\sup_{t \in [a,b]} |x - t| = \max \{ x - a, b - x \} = \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right|,
\]

(2.1) follows from (2.3) and (2.2).

Now suppose that (2.1) holds with a constant \( c > 0 \), that is,

\[
|J(x)| \leq \left[ c (b - a) + \left| x - \frac{a + b}{2} \right| \right] \int_a^b (f)
\]

for all \( x \in [a, b] \). For \( x = (a + b)/2 \), we get

\[
\left| \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} (b - a) \right| \leq c (b - a) \int_a^b (f). \tag{2.4}
\]

Define \( f : [a, b] \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0 & \text{if } x = a \\
1 & \text{if } x \in (a, b) \\
0 & \text{if } x = b.
\end{cases}
\]

Then \( f \) is of bounded variation on \([a, b]\) and

\[
\int_a^b f(x) \, dx = b - a, \quad \int_a^b (f) = 2.
\]

For this choice of \( f \), (2.4) provides

\[
b - a \leq 2c (b - a)
\]

or \( c \geq 1/2 \), concluding the proof. \( \square \)
Remark 1  

a) The choice \( x = b \) supplies the “left rectangle” inequality
\[
\left| \int_a^b f(x) \, dx - f(a) (b-a) \right| \leq (b-a) \int_a^b f(x) \, dx.
\]

b) Setting \( x = a \) yields the “right rectangle” inequality
\[
\left| \int_a^b f(x) \, dx - f(b) (b-a) \right| \leq (b-a) \int_a^b f(x) \, dx.
\]

c) For \( x = (a + b)/2 \) we obtain the known “trapezoid” inequality (1.1). This is the best possible inequality we can derive from (2.1) in the sense that the constant \( 1/2 \) is best possible.

Further standard assumptions about \( f \) lead to useful corollaries.

Corollary 1 Suppose \( f \in C^1[a,b] \). Then
\[
|J(x)| \leq \left\{ \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right\} \|f'\|_1
\]
for all \( x \in [a,b] \). Here as subsequently \( \|\cdot\|_1 \) is the \( L_1 \)-norm
\[
\|f'\|_1 := \int_a^b |f'(t)| \, dt.
\]

Corollary 2 Let \( f : [a,b] \to \mathbb{R} \) be a Lipschitzian mapping with the constant \( L > 0 \). Then
\[
|J(x)| \leq \left[ \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] (b-a) L
\]
for all \( x \in [a,b] \).
Proof. As $f$ is $L$–Lipschitzian on $[a, b]$, it is also of bounded variation. If $\text{Div}[a, b]$ denotes the family of divisions on $[a, b]$, then

$$\begin{align*}
\sup_{a}^{b} (f) &= \sup_{I_n \in \text{Div}[a, b]} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \\
&\leq L \sup_{I_n \in \text{Div}[a, b]} |x_{i+1} - x_i| \\
&= (b - a) L,
\end{align*}$$

and the desired result is proved. $\square$

Corollary 3 Let $f : [a, b] \to \mathbb{R}$ be a monotone mapping on $[a, b]$. Then

$$|J(x)| \leq \left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] |f(b) - f(a)|$$

for all $x \in [a, b]$.

For $f : [a, b] \to \mathbb{R}$ convex on $[a, b]$, we have the Hermite–Hadamard inequality

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$

The above results enable us to place bounds on the difference between the two sides of the second inequality. Thus if $f$ is convex and of bounded variation on $[a, b]$, (1.1) provides

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \leq \frac{1}{2} \sqrt{\int_{a}^{b} (f) \, dt}.$$

If $f$ is convex and Lipschitzian with the constant $L$ on $[a, b]$, then Corollary 2.3 yields

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \leq \frac{1}{2} (b - a) L.$$
If \( f \) is convex and monotonic on \([a, b]\), then by Corollary 2.4
\[
0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} |f(b) - f(a)|.
\]
Finally, if \( f \in C^1([a, b]) \) and convex, then by Corollary 2.2
\[
0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} \|f'\|_1.
\]

3. Applications to Quadrature Formulae

We now introduce the intermediate points \( \xi_i \in [x_i, x_{i+1}] \) \((i = 0, \ldots, n - 1)\) in the division \( I_n \) of \([a, b]\) and define
\[
T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} ((\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})).
\]
We have the following result concerning the approximation by \( T_P \) of the integral \( \int_a^b f(x) dx \).

Theorem 2 Let \( f : [a, b] \to \mathbb{R} \) be of bounded variation on \([a, b]\). Then
\[
\int_a^b f(x) dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi), \quad (3.1)
\]
with remainder term satisfying
\[
|R_P(f, I_n, \xi)| \leq \left[ \frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \int_a^b (f) \leq \nu(h) \int_a^b (f), \quad (3.2)
\]
The constant \( 1/2 \) is best possible.

Proof. Application of Theorem 1 to the intervals \([x_i, x_{i+1}]\) \((i = 0, \ldots, n - 1)\) gives
\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - [f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i)] \right|.
\]
for all \( i \in \{0, \ldots, n-1\} \).

By this and the generalized triangle inequality,

\[
|R_P (f, I_n, \xi)| \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t) \, dt - \left[ f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i) \right] \right|
\]

\[
\leq \sum_{i=0}^{n-1} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \left( \int_{x_i}^{x_{i+1}} f(t) \, dt \right)
\]

\[
\leq \max_{0 \leq i < n} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} f(t) \, dt \right)
\]

\[
\leq \left[ \frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \left( \int_{a}^{b} f(x) \, dx \right)
\]

and the first inequality in (3.2) is proved.

For the second, we observe that

\[
\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2} h_i, \quad i = 0, \ldots, n-1
\]

so that

\[
\max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2} \nu(h),
\]

proving the theorem. \( \square \)

**Remark 2**  

a) Choosing \( \xi_i = x_{i+1} \) \( i = 0, \ldots, n-1 \) provides

\[
\int_{a}^{b} f(x) \, dx = D_L (f, I_n) + R_L (f, I_n).
\]
Here \( D_L (f, I_n) \) is constructed from the left rectangle rule

\[
D_L (f, I_n) = \sum_{i=0}^{n-1} f (x_i) h_i
\]

and the remainder satisfies

\[
|R_L (f, I_n)| \leq \nu (h) \sqrt{\frac{b-a}{f}}.
\]

b) Taking \( \xi_i = x_i \) for \( i = 0, \ldots, n-1 \) gives

\[
\int_a^b f (x) \, dx = D_R (f, I_n) + R_R (f, I_n),
\]

where \( D_R (f, I_n) \) is built from the right rectangle rule

\[
D_R (f, I_n) = \sum_{i=0}^{n-1} f (x_{i+1}) h_i
\]

and the remainder term satisfies

\[
|R_R (f, I_n)| \leq \nu (h) \sqrt{\frac{b-a}{f}}.
\]

c) Finally, if we choose \( \xi_i = (x_i + x_{i+1})/2 \), we get (1.2) with (1.3) and (1.4).

**Corollary 4** Let \( f : [a, b] \to \mathbb{R} \) be Lipschitzian with constant \( L > 0 \). Then we have (3.1) and the remainder satisfies

\[
|R_T (f, I_n, \xi)| \leq L \left[ \nu (h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \leq L \nu (h).
\]

**Corollary 5** Let \( f : [a, b] \to \mathbb{R} \) be monotone on \([a, b]\). Then we have the quadrature formula (3.1) and the remainder satisfies

\[
|R_T (f, I_n, \xi)| \leq \left[ \frac{1}{2} \nu (h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] |f (b) - f (a)|
\]

\[
\leq \nu (h) |f (b) - f (a)|.
\]
4. Applications to Probability

Proposition 1  Let \( f : [a, b] \to \mathbb{R} \) be a probability density function of bounded variation on \([a, b]\) and \( F : [a, b] \to \mathbb{R} \) the corresponding distribution function

\[
F(x) = \int_a^x f(t) \, dt, \quad x \in [a, b].
\]

Then

\[
|F(x) - [f(a)(y-a) + f(x)(x-y)]| \leq \left[ \frac{1}{2}(x-a) + \left| y - \frac{a+x}{2} \right| \right] \frac{x}{a}(f) \quad (4.1)
\]

for all \( a \leq y \leq x \). In particular, choosing \( y = (a+x)/2 \) gives

\[
\left| F(x) - \frac{f(a) + f(x)}{2}(x-a) \right| \leq \frac{1}{2}(x-a) \frac{x}{a}(f) \quad (4.2)
\]

for all \( x \in [a, b] \). The constant \( 1/2 \) in (4.1) and (4.2) is best possible.

Proof. The result is immediate from Theorem 1. \( \square \)

The following approximation holds for the expectation of a random variable.

Proposition 2  Let \( X \) be a random variable having distribution function \( F \) and expectation \( E(X) \). Then

\[
\left| E(X) - \sum_{i=0}^{n-1} F(x_i)(\xi_{i+1} - \xi_i) - \xi_{n-1} \right| \leq \frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| . \quad (4.3)
\]

Proof. We apply Theorem 2 to \( F \) to get

\[
\left| \int_a^b F(t) \, dt - \sum_{i=0}^{n-1} F(x_i)(\xi_{i+1} - \xi_i) - \sum_{i=0}^{n-1} F(x_{i+1})(x_{i+1} - \xi_i) \right| \quad (4.4)
\]

\[
\leq \left[ \frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \frac{b}{a}(F). \]

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But
\[
\int_a^b (F) = F(b) - F(a) = 1
\]
and
\[
\int_a^b F(t) \, dt = tF(t)\bigg|_a^b - \int_a^b tf(t) \, dt = bF(b) - aF(a) - E(X) = b - E(X).
\]

By (4.4),
\[
\left| b - E(X) - F(a)(\xi_0 - a) - \sum_{i=1}^{n-1} F(x_i)(\xi_i - x_i) - \sum_{i=0}^{n-2} F(x_{i+1})(x_{i+1} - \xi_i) - F(b)(b - \xi_{n-1}) \right| \\
\leq \frac{1}{2} \nu(h) + \max_{0 \leq i < n} |\xi_i - \frac{x_i + x_{i+1}}{2}|
\]
or
\[
\left| -E(X) - \sum_{i=1}^{n-1} F(x_i)(\xi_i - \xi_{i-1}) + \xi_{n-1} \right| \leq \frac{1}{2} \nu(f) + \max_{0 \leq i < n} |\xi_i - \frac{x_i + x_{i+1}}{2}|
\]
and the proposition is proved. □

**Remark 3**  

a) Suppose the division is reduced to the endpoints, that is, \(a = x_0 < x_1 = b\) and \(\xi_1 = \xi \in [a, b]\). Then by (4.3)
\[
|E(X) - \xi| \leq \frac{1}{2} (b - a) + \left| \xi - \frac{a + b}{2} \right|
\]
for all \(\xi \in [a, b]\).
b) Suppose $a = x_0 < x < x_2 = b$ and $\xi \in [a, x], \mu \in [x, b]$. Then by (4.3)
\[
|E(X) - F(x)(\xi - \mu) - \mu| \\
\leq \frac{1}{2} \max \{|x - a|, |b - x|\} + \max \left\{\left|\xi - \frac{a + x}{2}\right|, \left|\mu - \frac{x + b}{2}\right|\right\} \\
= \frac{1}{2} (b - a) + \left|x - \frac{a + b}{2}\right| + \max \left\{\left|\xi - \frac{a + x}{2}\right|, \left|\mu - \frac{x - b}{2}\right|\right\}
\]
for all $a \leq \xi \leq x \leq \mu \leq b$.
In particular, if $\xi = (a + x)/2$ and $\mu = (x + b)/2$, then
\[
|E(X) - \frac{1}{2} F(x)(a - b) - \frac{x + b}{2}| \leq \frac{1}{2} (b - a) + \left|x - \frac{a + b}{2}\right|
\]
for all $x \in [a, b]$.

5. Applications to Special Means
We now derive some results for various well-known means. For $a, b \geq 0$ we have the arithmetic mean
\[
A = A(a, b) := (a + b)/2
\]
and the geometric mean
\[
G = G(a, b) := \sqrt{ab}.
\]
For $a, b > 0$ we have the harmonic mean
\[
H = H(a, b) := 2/ (a^{-1} + b^{-1}) ,
\]
the logarithmic mean
\[
L = L(a, b) := \left\{\begin{array}{ll}
a & \text{if } a = b \\
\frac{b - a}{\ln b - \ln a} & \text{if } a \neq b,
\end{array}\right.
\]
the identric mean
\[
\]
and for $p \in \mathbb{R} \setminus \{-1, 0\}$, the $p$–logarithmic mean
\[
L_p = L_p(a, b) := \begin{cases} 
\left\{ \frac{b^{p+1} - a^{p+1}}{(p+1)(b - a)} \right\}^{1/p} & \text{if } a \neq b; \\
\frac{a}{b} & \text{if } a = b.
\end{cases}
\]

It is well–known that with $L_{-1} := L$ and $L_0 := I$, the net $(L_p)$ is monotone nondecreasing in $p \in \mathbb{R}$. In particular, we have the inequalities
\[H \leq G \leq L \leq I \leq A.
\]

In what follows we establish some rather more involved inequalities for the above means by the use of (2.1), which we express in the equivalent form
\[
\left| \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{bf(b) - af(a)}{b - a} + x \cdot \frac{f(b) - f(a)}{b - a} \right| 
\leq \left[ \frac{1}{2} (b - a) + |x - A| \right] \frac{1}{b - a} \sqrt{\int_a^b (f)}.
\]

Define $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ by $f(x) = x^p$, $p \in \mathbb{R} \setminus \{-1, 0\}$. Then
\[
\frac{1}{b - a} \int_a^b f(t) \, dt = L_p^p(a, b),
\]
\[
\frac{bf(b) - af(a)}{b - a} = (p + 1) L_p^p(a, b),
\]
\[
\frac{f(b) - f(a)}{b - a} = pL_{p-1}^p(a, b),
\]
We deduce from (5.1) that

\[ |L_p^p - (p+1)L_p^p + pxL_p^{p-1}| \leq \left[ \frac{1}{2}(b-a) + |x-A| \right] |L_p^{p-1}|, \]

which is equivalent to

\[ |xL_p^{p-1} - L_p^p| \leq \left[ \frac{1}{2}(b-a) + |x-A| \right] L_p^{p-1}, \quad x \in [a,b]. \]

The choice \( x = A \) yields

\[ |AL_p^{p-1} - L_p^p| \leq \frac{1}{2}(b-a)L_p^{p-1}. \]

If instead we define \( f : [a,b] \subset (0,\infty) \to \mathbb{R} \) by \( f(x) = 1/x \), then

\[ \frac{1}{b-a} \int_a^b f(t) \, dt = L^{-1}(a,b), \]

\[ \frac{bf(b) - af(a)}{b-a} = 0, \]

\[ \frac{f(b) - f(a)}{b-a} = -G^{-2}(a,b) , \]

\[ \frac{1}{b-a} \int_a^b f(t) \, dt = G^{-2}(a,b). \]

From (5.1), we deduce that

\[ |L^{-1} - xG^{-2}| \leq \left[ \frac{1}{2}(b-a) + |x-A| \right] G^{-2}. \]
or equivalently

$$|xL - G^2| \leq \frac{1}{2} [(b - a) + |x - A|] L, \quad x \in [a, b].$$

Choosing $x = A$, we get

$$0 \leq AL - G^2 \leq \frac{1}{2} (b - a).$$

Finally, define $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ by $f(x) = \ln x$, so that

$$\frac{1}{b - a} \int_a^b f(t) \, dt = \ln I(a,b),$$

$$\frac{bf(b) - af(a)}{b - a} = \ln I(a,b) + 1,$$

$$\frac{f(b) - f(a)}{b - a} = L^{-1}(a,b),$$

$$\frac{1}{b - a} \int_a^b |f'(t)| \, dt = L^{-1}(a,b).$$

From (5.1), we deduce that

$$|x - L| \leq \frac{1}{2} (b - a) + |x - A|, \quad x \in [a, b].$$

With $x = A$, we get

$$0 \leq A - L \leq \frac{1}{2} (b - a).$$

### 6. Application to Euler’s Beta Function

Let $\beta$ be the Euler beta function given by

$$\beta(p, q) := \int_0^1 t^{p-1} (1 - t)^{q-1} \, dt, \quad p, q > 0.$$
Proposition 3. If $p, q > 1$, then

$$\beta(p, q) = T(p, q, I_n, \xi) + R(p, q, I_n, \xi),$$

where

$$T(p, q, I_n, \xi) = \sum_{i=0}^{n-1} \left[ (\xi_i - x_i) x_i^{p-1} (1 - x_i)^{q-1} + (x_{i+1} - \xi_i) x_{i+1}^{p-1} (1 - x_{i+1})^{q-1} \right]$$

and the remainder $R(p, q, I_n, \xi)$ satisfies

$$|R| \leq \left[ \frac{1}{2} \nu(h) + \max \left| \frac{\xi_i - x_i + x_{i+1}}{2} \right| \right] \max (p - 1, q - 1) \beta(p - 1, q - 1).$$

Proof. For $p, q > 1$ define $f_{p,q} : (0, 1) \to \mathbb{R}$ by

$$f_{p,q}(t) = t^{p-1} (1 - t)^{q-1}.$$ 

We have

$$f'_{p,q}(t) = [(p + q - 2) t - q + 1] t^{p-2} (1 - t)^{q-2},$$

so that

$$\frac{1}{0} (f_{p,q}) = \int_{0}^{1} |f'_{p,q}(t)| dt$$

$$\leq \int_{0}^{1} |(p + q - 2) t - q + 1| t^{p-2} (1 - t)^{q-2} dt$$

$$\leq \max (q - 1, p - 1) \int_{0}^{1} t^{p-2} (1 - t)^{q-2} dt$$

$$= \max (q - 1, p - 1) \beta(p - 1, q - 1)$$

and the proposition is proved. \qed

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Remark 4 The choice \( \xi_i = (x_i + x_{i+1})/2 \) yields

\[
\beta(p, q) = T(p, q, I_n) + R(p, q, I_n)
\]

where

\[
T(p, q, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} \left[ x_i^{p-1} (1 - x_i)^{q-1} + x_{i+1}^{p-1} (1 - x_{i+1})^{q-1} \right] h_i
\]

corresponds to the trapezoid rule and the remainder satisfies

\[
|R(p, q, I_n)| \leq \frac{1}{2} \nu(h) \max(p - 1, q - 1) \beta(p - 1, q - 1)
\]

for all \( p, q > 1 \).

References


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