Zeros of $\zeta''(s)$ & $\zeta'''(s)$ in $\sigma < \frac{1}{2}$

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Abstract

There is only one pair of non-real zeros of $\zeta''(s)$, and of $\zeta'''(s)$, in the left half-plane. The Riemann Hypothesis implies that $\zeta''(s)$ and $\zeta'''(s)$ have no zeros in the strip $0 \leq \Re s < \frac{1}{2}$.

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1. Introduction

The Riemann zeta-function defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\sigma > 1)$$

(as usual we write $s = \sigma + it; \sigma, t \in \mathbb{R}$), can be analytically continued to the whole complex plane, with a simple pole at $s = 1$, and satisfies the functional equation

$$\zeta(1 - s) = 2(2\pi)^{-s} \cos \left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s).$$

From (2) it is seen that $\zeta(-2n) = 0, \forall n \in \mathbb{Z}^+$ (trivial zeros of $\zeta$). From Hadamard’s theory of entire functions it follows that $\zeta(s)$ also has infinitely many (nontrivial) zeros in the strip $0 < \sigma < 1$. The nontrivial zeros are situated symmetrically with respect to the real axis and also with respect to the line $\sigma = \frac{1}{2}$. Applying the argument principle, von Mangoldt proved that the number of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ is

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T),$$

as $T \to \infty$. Riemann’s yet unproved assertion that all of these zeros lie on the critical line $\sigma = \frac{1}{2}$ is known as the Riemann Hypothesis (RH). For the fundamentals of the theory of $\zeta(s)$ we refer the reader to Davenport’s book [3].
The origin of our topic is Speiser’s proof [6] that the Riemann Hypothesis is equivalent to \( \zeta'(s) \) having no zeros in \( 0 < \sigma < \frac{1}{2} \). In a comprehensive article on the zeros of derivatives of \( \zeta(s) \), Levinson and Montgomery [4] gave a different proof of this and that \( \zeta'(s) \) has only real zeros in the closed left half-plane, vanishing exactly once in the interval \((-2n - 2, -2n)\) for \( n \geq 1 \) (these are the zeros between the trivial zeros of \( \zeta \) guaranteed by Rolle’s theorem). Moreover they showed that for any \( k \geq 1 \), \( \zeta^{(k)}(s) \) has at most a finite number of nonreal zeros in \( \sigma < \frac{1}{2} \) as a consequence of RH. Spira [7] calculated the zeros of \( \zeta' \) and \( \zeta'' \) in the rectangle \(-1 \leq \sigma \leq 5, |t| \leq 100\), and found out that \( \zeta''(s) \neq 0 \) in \( 0 \leq \sigma \leq \frac{1}{2}, |t| \leq 100 \). However, Spira also found that \( \zeta'' \) has zeros at \(-0.355084 \pm i \cdot 3.59083\) (to be denoted as \( b_0 \) and \( \overline{b}_0 \) below).

Berndt [2] showed that the number of nonreal zeros of \( \zeta^{(k)}(s) \) with imaginary parts in \([0, T]\) is \( \frac{T}{2\pi} \log T - \frac{1 + \log 4\pi}{2\pi} T + O(\log T) \). For each \( k \geq 0 \), the nonreal zeros of \( \zeta^{(k)}(s) \) all lie in a strip \( \alpha_k < \sigma < \sigma_k \). The existence of \( \alpha_k \) was deduced by Spira [8]. That \( \zeta(s) \neq 0 \) in the region \( \sigma \geq 1 - \frac{c}{\log T}, t \geq 2 \) (in fact the very first zero of \( \zeta(s) \) is at \( \frac{1}{2} + i \cdot 14.134.. \), and the first \( 1.5 \cdot 10^9 \) zeros of \( \zeta(s) \) have all been verified in [5] to lie on the critical line) implies the prime number theorem. Titchmarsh [9, Theorem 11.5c] proved \( \sigma_1 < 3 \). Later Spira [7] calculated that \( \sigma_2 = 4.98\ldots, \sigma_3 = 6.01\ldots, \sigma_{10} = 13.68\), and in general \( \sigma_k = \frac{2}{k} + 2 \) for \( k \geq 3 \) is acceptable. Verma and Kaur [10] have improved the last estimate to \( \sigma_k = ak + 2 \) for \( k \geq 3 \) with \( a = 1.13\ldots \)

In this paper, we shall be concerned with the zeros of \( \zeta''(s) \) and \( \zeta'''(s) \) lying to the left of the critical line. Our results for the left half-plane are unconditional (i.e. without assuming RH), since here \( \zeta(s) \) can be expressed via the functional equation in terms of its values in \( \sigma > 1 \), but to get results for the strip \( 0 < \sigma < \frac{1}{2} \) we assume RH. Most of our results appeared in [11] which contained only the proof of Theorem 1 fully.

In our calculations we will repeatedly use the well-known formula

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{m} \frac{a_k}{n + \alpha_k} \right) = -\sum_{k=1}^{m} a_k \psi(1 + \alpha_k), \quad (a_k, \alpha_k \in \mathbb{C})
\]

where \( \psi = \frac{d}{d\log z} \) is the digamma function.
2. \( \zeta'' \) to the left of the critical line

Theorem 1. The Riemann Hypothesis implies that \( \zeta''(s) \) has no zeros in the strip \( 0 \leq \sigma < \frac{1}{2} \).

Proof. Let us denote the real zeros of \( \zeta' \) as \(-a_n, n \geq 1\), where \( a_n \in (2n, 2n+2)\). A nonreal zero of \( \zeta' \) will be represented as \( \rho_i = \beta_i + i\gamma_i \). By what was recounted above, \( \frac{1}{2} \leq \beta_i < 3 \) for all \( \rho_i \) (the lower-bound is upon RH). Since \( \Re \zeta'(s) < 0 \) on \( \sigma = \frac{1}{2} \) except when \( \zeta(s) = 0 \), one has \( \beta_i = \frac{1}{2} \) only at a possible multiple zero of \( \zeta(s) \) (see [4]). We start with the partial fraction representation

\[
\frac{\zeta''}{\zeta'}(s) = \frac{\zeta''}{\zeta'}(0) - 2 - \frac{2}{s-1} + \sum_{\rho_1} \left( \frac{1}{s-\rho_1} + \frac{1}{\rho_1} \right) + \sum_n \left( \frac{1}{s+a_n} - \frac{1}{a_n} \right),
\]

(3)

which follows from Hadamard factorization. Taking real parts in (3), we have

\[
\Re \frac{\zeta''}{\zeta'}(s) = \frac{\zeta''}{\zeta'}(0) - 2 + \frac{2(1-\sigma)}{|s-1|^2} + \sum_{\rho_1} \Re \frac{1}{s-\rho_1} + \sum_{\rho_1} \frac{1}{\rho_1}
+ \sum_n \left( \frac{\sigma + a_n}{|s+a_n|^2} - \frac{1}{a_n} \right),
\]

(4)

since \( \zeta'(|\sigma|) = 0 \) as well. We should first like to put a bound on \( \sum_{\rho_1} \frac{1}{\rho_1} \) (in this series it is understood that the terms from \( \rho_1 \) and \( \overline{\rho_1} \) are grouped together). At \( s = 6 \), Eq. (4) reads

\[
\frac{\zeta''}{\zeta'}(6) = \frac{\zeta''}{\zeta'}(0) - \frac{12}{5} + \sum_{\rho_1} \frac{6 - \beta_i}{(6 - \beta_i)^2 + \gamma_i^2} + \sum_{\rho_1} \frac{\beta_i}{\beta_i^2 + \gamma_i^2}
- \sum_n \frac{6}{a_n(a_n+6)}
\]

(5)

It is known that \( \frac{\zeta''}{\zeta'}(0) = 2.183.. \) (see [1]), and \( \frac{\zeta''}{\zeta'}(6) = -0.773.. \) Also

\[
\sum_n \frac{6}{a_n(a_n+6)} < \sum_{n=1}^{\infty} \frac{6}{2n(2n+6)} = \frac{11}{12}.
\]

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Since the least $|\gamma_1|$ is 23.3.. (see [7]), for all $\rho_1$ we have
\[
\frac{6 - \beta_1}{(6 - \beta_1)^2 + \gamma_1^2} > \frac{\beta_1}{\beta_1^2 + \gamma_1^2}.
\]
Plugging all these in (5), it follows that
\[
\sum_{\rho_1} \frac{1}{\rho_1} < 0.185
\]
(6)
(from Spira’s list of $\rho_1$ with $|\gamma_1| < 100$ one calculates $\sum \frac{1}{\rho_1} > 0.0249$).

We now examine the value of $\Re\zeta''(s)$ in the region $0 \leq \sigma \leq \frac{1}{2}$, $|t| \geq 100$. If ever a zero of $\zeta'$ exists on the critical line, this region is to be modified by deleting an arbitrarily small neighbourhood around such a zero. For any $s$ in our region, $\frac{2(1 - \sigma)}{|s - 1|^2} < \frac{1}{5000}$ and $\Re\frac{1}{s - \rho_1} < 0$ for all $\rho_1$ (on RH), and also
\[
\sum_n \left( \frac{\sigma + a_n}{|s + a_n|^2} - \frac{1}{a_n} \right) \leq \sum_n \frac{-10^4}{a_n((a_n + \frac{1}{2})^2 + 10^4)} \\
< \sum_{n=2}^\infty \frac{-10^4}{2n((2n + \frac{1}{2})^2 + 10^4)} \\
< \frac{1}{2} \left( 1 + \psi(1) - \Re\psi\left(\frac{5}{4} + 50i\right) \right) + \frac{\Im\psi\left(\frac{5}{4} + 50i\right)}{400} \\
< -1.74.
\]
Together with (6), these estimates used in (4) give $\Re\zeta''(s) < -1.37$ at all points of our region.

Notice that $\zeta''(s)$ can be zero on the critical line only at a multiple zero (of at least third order) of $\zeta(s)$ if ever this exists.

**Theorem 2.** (unconditional) There is only one pair of nonreal zeros of $\zeta''(s)$ in the left half-plane.
To prove Theorem 2 we shall consider the change in the argument of \( \frac{\zeta'}{\zeta}(s) \) as \( s \) goes around the rectangle \( R \) with corners at \( \pm iN, \sigma_N \pm iN \), where \( \sigma_N = -2N - 2 \) with an arbitrarily large \( N \in \mathbb{N} \). The reason behind this choice of \( \sigma_N \) will be clear after the following lemma.

**Lemma 1.** \(-a_n = -2n - 2 + \frac{1}{\log n} + O\left(\frac{1}{\log^2 n}\right)\), as \( n \to \infty \).

**Proof.** Differentiating the functional equation (2) we have

\[
\zeta'(1 - s) = \zeta(1 - s) \left[ \log 2\pi + \frac{\pi}{2} \tan \frac{\pi s}{2} - \psi(s) - \frac{\zeta'}{\zeta}(s) \right],
\]

so we see that \( \zeta'(1 - \sigma) = 0 \) with \( \sigma > 1 \) if

\[
\log 2\pi + \frac{\pi}{2} \tan \frac{\pi \sigma}{2} - \psi(\sigma) - \frac{\zeta'}{\zeta}(\sigma) = 0.
\] (8)

We are interested in the situation when \( \sigma \to \infty \), in which case we use

\[
\psi(\sigma) = \log \sigma - \frac{1}{2\sigma} + O\left(\frac{1}{\sigma^2}\right),
\]

\[
\frac{\zeta'}{\zeta}(\sigma) = -\left(\frac{\log 2}{2\sigma} + \frac{\log 3}{3\sigma} + \frac{\log 2}{4\sigma} + \ldots\right) = O\left(\frac{1}{\sigma^2}\right).
\] (10)

Thus as \( \sigma \to \infty \), (9) becomes

\[
\frac{\pi}{2} \tan \frac{\pi \sigma}{2} = \log \frac{\sigma}{2\pi} - \frac{1}{2\sigma} + O\left(\frac{1}{\sigma^2}\right).
\]

Since the right-hand side tends to \( \infty \), to maintain equality we must have \( \sigma \) tend to \( \infty \) through values close to and to the left of odd integers. So the negative zeros of \( \zeta' \) lie slightly to the right of negative even integers, i.e.

\[-a_n = -2n - 2 + \epsilon(n), \quad (\epsilon(n) > 0)\].

Carrying this out in more detail, taking \( \sigma = 2n + 3 - \epsilon(n) \), we have

\[
\frac{\pi}{2} \tan \frac{\pi \sigma}{2} = \frac{1}{\epsilon(n)} - \frac{\pi^2 \epsilon(n)}{12} + O(\epsilon^3(n)).
\]
Thus we find
\[ \epsilon(n) = \frac{1}{\log n} + O\left(\frac{1}{\log^2 n}\right), \]
and \( \epsilon(n) < 1 \) for \( n \geq 3 \). \( \square \)

Note that differentiation of (2) also gives
\[ \zeta'(-2k) = (-1)^k \pi (2\pi)^{-(2k+1)} (2k)! \zeta(2k + 1). \] (12)

Next we observe that \( \frac{\zeta''}{\zeta'}(-\sigma_N) < 0 \) for all sufficiently large \( N \). For, differentiating the functional equation twice, we get for \( k \geq 1 \),
\[ \zeta''(-2k) = (-1)^k \frac{(2k)!}{(2\pi)^{2k}} \left[ \zeta(2k + 1)(\log 2\pi - \psi(2k + 1)) - \zeta'(2k + 1) \right] \] (13)
and so we have
\[ \frac{\zeta''}{\zeta'}(-2k) = 2 \left[ \log 2\pi - \psi(2k + 1) - \frac{\zeta'}{\zeta}(2k + 1) \right] < 0, \quad (k \geq 3). \] (14)

**Proof of Theorem 2.** Inside \( R \) there are exactly \( N \) zeros of \( \zeta' \) (all real), so by Rolle’s theorem there must be at least \( N - 1 \) real zeros of \( \zeta'' \). We also know that there exist \( 2\kappa, \kappa \geq 1 \), nonreal zeros of \( \zeta'' \) inside \( R \). Call the number of zeros of \( \zeta^{(i)} \) in \( R \) as \( Z_i \). By the argument principle we have
\[ \frac{1}{2\pi} \Delta_R \text{arg} \frac{\zeta''}{\zeta'}(s) = Z_2 - Z_1 \geq N - 1 + 2\kappa - N = 2\kappa - 1. \]

If it is shown that \( \text{arg} \frac{\zeta''}{\zeta'}(s) \) changes by \( 2\pi \) as \( s \) makes one counterclockwise tour of \( R \), then Theorem 2 is proved. It would also follow that between consecutive negative zeros of \( \zeta' \), \( \zeta'' \) vanishes exactly once.

Equation (4) may be rewritten as
\[ \Re \frac{\zeta''}{\zeta'}(\sigma + it) = K + \frac{2(1 - \sigma)}{(1 - \sigma)^2 + t^2} + \sum_n \left( \frac{(\sigma + a_n)}{(\sigma + a_n)^2 + t^2} - \frac{1}{a_n} \right) \]
\[ + \sum_{\rho_1} \frac{\sigma - \beta_1}{(\sigma - \beta_1)^2 + (\gamma - t)^2}, \] (15)
where \( K = \zeta''(0) - 2 + \sum_{\rho_1} \frac{1}{\rho_1} \) and \( 0.185 < K < 0.368 \).
First consider the left edge of $R$ where $\sigma = \sigma_N = -2N - 2$, $|t| \leq N$. Here

$$
\frac{2(1 - \sigma)}{(1 - \sigma)^2 + t^2} = O\left(\frac{1}{N}\right),
$$

and $-2N - 5 \leq \sigma_N - \beta_1 \leq -2N - 2$, so that (writing $\sum_{\rho_1}$ for the last term of (15))

$$
-(2N + 5) \sum_{\gamma_1 \leq 2N + 1} \frac{1}{(2N + 2)^2 + (\gamma_1 - t)^2} < \sum_{\rho_1} < -(2N + 2) \sum_{\gamma_1 \leq 2N + 1} \frac{1}{(2N + 5)^2 + (\gamma_1 - t)^2}
$$

$$
-(2N + 5) \sum_{|\gamma_1 - t| < 2N + 2} \frac{1}{(2N + 2)^2} + \sum_{|\gamma_1 - t| > 2N + 5} \frac{1}{(\gamma_1 - t)^2}
$$

$$
< \sum_{\rho_1} < -(2N + 2) \left( \sum_{|\gamma_1 - t| < 2N + 5} \frac{1}{2(2N + 5)^2} + \sum_{|\gamma_1 - t| > 2N + 5} \frac{1}{2(\gamma_1 - t)^2} \right)
$$

The sums over $\gamma_1$ are evaluated in a standard way using the result of Berndt mentioned in the introduction, giving

$$
-\frac{2}{\pi} \log N \lesssim \sum_{\rho_1} \frac{\sigma - \beta_1}{(\sigma - \beta_1)^2 + (\gamma_1 - t)^2} \lesssim -\frac{1}{\pi} \log N.
$$

Now consider the sum over $n$ in (15) for $\sigma_N \leq \sigma < 0$, splitting it into two parts: $\sigma + a_n \leq 0$ (the finite part) and $\sigma + a_n > 0$ (the infinite part). The finite part is negative and attains its maximum at $|t| = N$. We have

$$
\sum_{a_n \leq -\sigma} \left( \frac{(\sigma + a_n)}{(\sigma + a_n)^2 + N^2} - \frac{1}{a_n} \right) \leq -\sum_{a_n \leq -\sigma} \frac{1}{a_n} + O(1)
$$

$$
\leq -\sum_{n < (\sigma/2)} \frac{1}{2n + 2} + O(1)
$$

$$
= -\frac{1}{2} \log(1 - \frac{\sigma}{2}) + O(1)
$$

(In (18) the sums over $a_n$ are void if $-\sigma < a_1$, and the sum over $n$ is void if $\sigma > -2$. In these cases the $O(1)$-term takes care of things). Thus on the left edge of $R$ the finite part
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is always less than $-\frac{1}{2} \log N + O(1)$. On the left edge of $R$ the infinite part is maximum when $t = 0$, and then by Lemma 1

$$\sum_{n=N+1}^{\infty} \left( \frac{1}{\sigma_n + a_n} - \frac{1}{a_n} \right) < \sum_{n=N+1}^{\infty} \left( \frac{1}{2(n-N) - \frac{2}{\log N} - \frac{1}{2n+2}} \right)$$

$$= \frac{1}{2} \sum_{m=1}^{\infty} \left( \frac{1}{m - \frac{1}{\log N} - \frac{1}{m+N+1}} \right)$$

$$= \frac{1}{2} \left( \psi(N+2) - \psi(1 - \frac{1}{\log N}) \right)$$

$$\leq \frac{1}{2} \log N + O(1). \tag{19}$$

Adding up the results of (16)-(19) in (15) we have on the left edge of $R$

$$\Re \frac{\zeta'}{\zeta} (\sigma + it) \lesssim -\frac{1}{\pi} \log N \quad (|t| \leq N). \tag{20}$$

On $\sigma + iN, \sigma_N \leq \sigma < 0$ we rewrite (15) as

$$\Re \frac{\zeta'}{\zeta} (\sigma + iN) = K + \sum_{a_n > \sigma} + \sum_{a_n \leq \sigma} + \sum_{\rho_1} + O\left( \frac{1}{N} \right),$$

where the sum over $\rho_1$ takes negative values, and the finite sum was estimated in (18).

Now observe that for $\sigma < 0$

$$\sum_{a_n > \sigma} \left( \frac{\sigma + a_n}{(\sigma + a_n)^2 + N^2} - \frac{1}{a_n} \right)$$

$$= -(\sigma^2 + N^2) \sum_{a_n > \sigma} \frac{1}{a_n((\sigma + a_n)^2 + N^2)} - \sigma \sum_{a_n > \sigma} \frac{1}{(\sigma + a_n)^2 + N^2}$$

$$< -(\sigma^2 + N^2) \sum_{n=\lceil \frac{-\sigma}{2n+2} \rceil}^{\infty} \frac{1}{(2n+2)((\sigma + 2n + 2)^2 + N^2)} + O(1) - \sigma \sum_{n=0}^{\infty} \frac{1}{(2n)^2 + N^2}.$$

For $\sigma_N \leq \sigma < 0$,

$$- \sigma \sum_{n=0}^{\infty} \frac{1}{(2n)^2 + N^2} = O(1),$$

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and we calculate

\[-(\sigma^2 + N^2) \sum_{n=\lfloor \frac{\sigma}{2} \rfloor}^{\infty} \frac{1}{(2n + 2)[(\sigma + 2n + 2)^2 + N^2]} \]

\[\leq -\frac{(\sigma^2 + N^2)}{8} \sum_{m=1}^{\infty} \frac{1}{(m - \frac{\sigma}{2})(m^2 + (\frac{\sigma}{2})^2)} \]

\[= \sum_{m=1}^{\infty} \left( -\frac{1}{2m - \frac{\sigma}{2}} - \frac{\sigma - iN}{4N} \frac{1}{m + \frac{\sigma}{2}} + \frac{\sigma + iN}{4Ni} \frac{1}{m - \frac{\sigma}{2}} \right) \]

\[= \frac{1}{2} \psi(1 - \frac{\sigma}{2}) + \frac{\sigma}{2N} \Re\psi(1 + \frac{iN}{2}) - \frac{1}{2} \Re\psi(1 + \frac{iN}{2}). \]

So for \(\sigma_N \leq \sigma < 0\)

\[\sum_{a_n > -\sigma} \left( \frac{\sigma + a_n}{(\sigma + a_n)^2 + N^2} - \frac{1}{a_n} \right) < \frac{1}{2} \psi(1 - \frac{\sigma}{2}) - \frac{1}{2} \Re\psi(1 + \frac{iN}{2}) + O(1) \]

\[= \frac{1}{2} \log(1 - \frac{\sigma}{2}) - \frac{\log N}{2} + O(1). \quad (21)\]

Hence on \(\sigma \pm iN, \sigma_N \leq \sigma < 0\) we have

\[\Re\zeta''(\sigma + iN) < -\frac{1}{2} \log N + O(1). \quad (22)\]

It remains to consider the edge on the imaginary axis, \([-iN, iN]\). Here,

\[\Re\zeta''(it) = \zeta''(0) - 2 + \frac{2}{1 + t^2} + \sum_n \frac{-t^2}{a_n(a_n^2 + t^2)} + \sum_{\rho} \frac{1}{\rho} \]

\[+ \sum_{\gamma_1 > 0} \left( \frac{-\beta_1}{(\beta_1^2 + (\gamma_1 - t)^2)} + \frac{-\beta_1}{(\beta_1^2 + (\gamma_1 + t)^2)} \right), \quad (23)\]

\[\Im\zeta''(it) = \frac{2t}{1 + t^2} - \sum_n \frac{t}{a_n^2 + t^2} + \sum_{\gamma_1 > 0} \frac{2t(\gamma_1^2 - \beta_1^2 - t^2)}{(\beta_1^2 + (\gamma_1 - t)^2)(\beta_1^2 + (\gamma_1 + t)^2)}. \quad (24)\]

The sums over \(a_n\) can be bounded in a similar way to (7), but keeping in mind that \(2.6 < a_1 < 2.8, 4.8 < a_2 < 5, \) and \(2n + 1 < a_n < 2n + 2\) for \(n \geq 3\) (this can be verified...
from (8)) in order to get sharper inequalities that will allow us below to determine the signs of $\Re C_{\gamma_1}(it)$ and $\Im C_{\gamma_1}(it)$ at certain points. Writing

$$A(t) = -\frac{t^2}{2.8(2.8^2 + t^2)} - \frac{t^2}{5(5^2 + t^2)} + \sum_{n=1}^{3} \frac{t^2}{2n(2n^2 + t^2)}$$

$$B(t) = -\frac{t^2}{2.6(2.6^2 + t^2)} - \frac{t^2}{4.8(4.8^2 + t^2)} + \frac{t^2}{3(3^2 + t^2)} + \frac{t^2}{5(5^2 + t^2)}$$

we have

$$B(t) + \frac{1}{2}(\psi(\frac{3}{2}) - \Re \psi(\frac{3}{2} + \frac{it}{2})) < \sum_{n} \frac{-t^2}{a_n(a_n^2 + t^2)} < A(t) + \frac{1}{2}(\psi(1) - \Re \psi(1 + \frac{it}{2})). \tag{25}$$

Similarly, writing

$$C(t) = -\frac{t}{2.6^2 + t^2} - \frac{t}{4.8^2 + t^2} + \frac{t}{3^2 + t^2} + \frac{t}{5^2 + t^2}$$

$$D(t) = -\frac{t}{2.8^2 + t^2} - \frac{t}{5^2 + t^2} - \frac{t}{4^2 + t^2} - \frac{t}{6^2 + t^2}$$

we have

$$C(t) - \frac{1}{2}(\psi(\frac{3}{2} + \frac{it}{2})) < \sum_{n} \frac{-t}{a_n^2 + t^2} < D(t) - \frac{1}{2}(\psi(1 + \frac{it}{2})). \tag{26}$$

Using (25) and (6) in (23), where taking 0 as an upper bound for the sum over $\gamma_1 > 0$, it is seen that for $t > 23$, $\Re C_{\gamma_1}(it) < 0$. In (23) we combine the sums over $\rho_1$ and $\gamma_1 > 0$ as

$$\sum_{\gamma_1 > 0} \frac{2t^2\beta_1(\beta_1^2 - 3\gamma_1^2 + t^2)}{(\beta_1^2 + \gamma_1^2)(\beta_1^2 + (\gamma_1 - t)^2)(\beta_1^2 + (\gamma_1 + t)^2)},$$

and we see that for $|t| < 40$ each term is negative (since $\gamma_1 > 23.298$ and $\beta_1 < 3 (\lceil \gamma_1 \rceil)$). Also, the derivative of a term of the sum over $\gamma_1$ in (23) is

$$\frac{-4t\beta_1[-\beta^4 + (\gamma^2 - t^2)(2\beta^2 + t^2 + 3\gamma^2)]}{[\beta^2 + (\gamma_1 - t)^2][\beta_1^2 + (\gamma_1 + t)^2]}.$$
which is negative for \( t \in [0, 23] \). So for \( t \in [0, 23] \) all the terms in the right-hand side of (23) are decreasing functions of \( t \). Hence \( \Re \frac{\psi''(i)}{\zeta}(t) = 0 \) at only one pair of conjugate points on the imaginary axis. As for \( \Im \frac{\psi''(i)}{\zeta}(t) \), the sum over \( \rho_1 \) in (24) is positive for \( 0 < t \leq 23 \). When \( t \rightarrow 0^\pm \) we have \( \Im \frac{\psi''(i)}{\zeta}(t) \rightarrow 0^\pm \) \( \left( \lim_{t \rightarrow 0^\pm} \frac{\Im \psi(1 + \frac{it}{2})}{t} = \frac{\pi^2}{12} \right. \) and

\[
\lim_{t \rightarrow 0^\pm} \frac{\Im \psi(\frac{i}{2} + \frac{it}{2})}{t} = \frac{\pi^2}{2} - 4.
\]

From eqs. (23)-(26) we see that

\[
\Re \frac{\psi''(i)}{\zeta}(t) > K + 1 + B(1) + \frac{1}{2} [\psi(\frac{3}{2}) - \Re \psi(\frac{3}{2} + \frac{i}{2})] + \sum_{\gamma_1 > 0} \frac{4\beta_1}{(\beta_1^2 + \gamma_1^2)}
\]

\[
= \frac{\zeta''(0)}{\zeta'}(0) - 1 + B(1) + \frac{1}{2} [\psi(\frac{3}{2}) - \Re \psi(\frac{3}{2} + \frac{i}{2})] - \sum_{\rho_1 > 0} \frac{1}{\rho_1} > 0,
\]

\[
\Im \frac{\psi''(i)}{\zeta}(t) > C(1) - \frac{1}{2} \Re \psi(\frac{3}{2} + \frac{i}{2}) > 0,
\]

\[
\Re \frac{\psi''(3.5i)}{\zeta}(0) - 2 + \frac{2}{13.25} + A(3.5) + \frac{1}{2} [\psi(1) - \Re \psi(1 + 1.75 i)] < 0,
\]

\[
\Im \frac{\psi''(3.5i)}{\zeta}(0) > \frac{7}{13.25} + C(3.5) - \frac{1}{2} \Re \psi(\frac{3}{2} + 1.75 i) + 0.0197 > 0,
\]

where 0.0197 is a lower bound for the first two terms of the sum over \( \gamma_1 > 0 \) in (24) coming from the first two zeros of \( \zeta' \) at approximately \( 2.46 \pm i \cdot 23.298 \), and \( 1.29 \pm i \cdot 31.71 \) ([7]).

As \( t \) increases from 3.5, \( \Im \frac{\psi''(i)}{\zeta}(it) \) may change sign, but \( \Re \frac{\psi''(i)}{\zeta}(it) \) will always be negative.

Thus as \( t \) moves up on the imaginary axis, the image curve of \( \frac{\psi''(i)}{\zeta}(it) \) includes just one counterclockwise loop around the origin and the change in arg \( \frac{\psi''(i)}{\zeta} \) is roughly \( 2\pi \). This completes the proof of Theorem 2.

\[ \square \]

The graphs of \( \frac{\psi''(i)}{\zeta(\zeta')} \) for \( |t| \leq 40 \) and \( k = 0, 1, 2, 3 \) are included at the end of this paper. These graphs were plotted by M. "Ozkan in his senior project, using the expressions from the Euler-Maclaurin sum formula.
\[-1^{k} \zeta^{(k)}(s) = \sum_{n=1}^{N-1} \frac{k}{n^s} + \frac{k}{2N^s} + N^{1-s} \sum_{j=0}^{k} C_{kj} \frac{k-j}{(s-1)^{j+1}} \]
\[+ \sum_{\nu=1}^{m} \left( \sum_{j=0}^{k} \binom{k}{j} \Pi_{\nu}^{(k-j)}(s)(-1)^{k-j} \log^{j} N \right) N^{1-s-2\nu} + R_k, \quad (27)\]

where
\[C_{kj} = \frac{k}{(k-j)!}, \quad \Pi_{\nu}^{(j)}(s) = \frac{B_{2\nu}}{(2\nu)!} ds^\nu \prod_{j=0}^{2\nu-2} (s+j),\]

and the error term $R_k$ is neglected in the computations.

In the proof of Theorem 2 one can also notice that if one starts from a point on the negative real axis where $\Re \zeta''(s) > 0$ and moves vertically away from the real axis, soon one hits a point where $\Re \zeta''(s) = 0$ and further away from the axis $\Re \zeta''(s) < 0$.

3. Zeros of $\zeta''(s)$ on the negative real axis

In order to proceed to the investigation of $\zeta''(s)$ some information on the negative zeros of $\zeta''(s)$ - which will be denoted by $-b_n$, $n \geq 1$ - is needed. The zeros of $\zeta''(s)$ in the right half-plane will be denoted by $\rho_2 = \beta_2 + i\gamma_2$. From Eq. (13) we see that $\zeta''(-2) < 0$, $\zeta''(-4) > 0$, and as of $k = 3$ the quantity in brackets in (13) will always be negative so that $\text{sgn} \left[ \zeta''(-2k) \right] = (-1)^{k+1}$, $(k \geq 3)$. Similar to (13) we have for $k \geq 1$

\[\zeta''(1-2k) = (-1)^k \frac{2\Gamma(2k)\zeta(2k)}{(2\pi)^{2k}} \left[ \log^2 2\pi - \left( \frac{\pi}{2} \right)^2 + \psi'(2k) + (\psi(2k))^2 + \frac{\zeta''}{\zeta}(2k) \right] \]
\[+ 2\psi(2k) \left[ \frac{\psi}{\zeta}(2k) \log 2\pi \right] - 2 \frac{\psi}{\zeta}(2k) \log 2\pi. \quad (28)\]

In (28), as $k$ increases, the term $(\psi(2k))^2$ will eventually dominate and the quantity in brackets will always be positive. This happens for $k \geq 16$. We find that

\[b_n \in \begin{cases} [3, 4] & n = 1 \\ [2n + 2, 2n + 3] & 2 \leq n \leq 13 \\ [2n + 3, 2n + 4] & n \geq 14. \end{cases} \quad (29)\]
Using MAPLE-V the first few negative zeros of $\zeta''$ are found to be
\begin{align*}
b_1 &= 3.595., \quad b_2 = 6.028., \quad b_3 = 8.278., \quad b_4 = 10.446., \\
b_5 &= 12.568., \quad b_6 = 14.662., \quad b_7 = 16.736., \quad b_8 = 18.798., \\
b_9 &= 20.849., \quad b_{10} = 22.933., \quad b_{11} = 24.931.
\end{align*}

Lemma 2. \(-b_n = -2n - 4 + \frac{2}{\log n} + O\left(\frac{1}{\log^2 n}\right), \text{ as } n \to \infty.\)

Proof. Differentiating (8) gives
\begin{equation}
-\frac{\zeta''}{\zeta}(1 - s) + \left(\frac{\zeta'}{\zeta}(1 - s)\right)^2 = \left(\frac{\pi}{2}\right)^2(1 + \tan^2 \frac{\pi s}{2}) - \psi'(s) - \left(\frac{\zeta'}{\zeta}(s)\right)'.
\end{equation}

We put $\zeta''(1 - \sigma) = 0$, $\sigma > 1$ and use (8) to write
\begin{equation}
\left(\log 2\pi + \frac{\pi}{2} \tan \frac{\pi \sigma}{2} - \psi(\sigma) - \frac{\zeta'}{\zeta}(\sigma)\right)^2 = \left(\frac{\pi}{2}\right)^2(1 + \tan^2 \frac{\pi \sigma}{2}) - \psi'(\sigma) - \left(\frac{\zeta'}{\zeta}(\sigma)\right)'.
\end{equation}

For large $\sigma$, using (10), (11) and
\begin{equation}
\psi'(\sigma) = \frac{1}{\sigma} + O\left(\frac{1}{\sigma^2}\right), \quad \frac{\zeta''}{\zeta}(\sigma) = O\left(\frac{1}{\sigma^2}\right)
\end{equation}
this is simplified to
\begin{equation}
\left[\log \frac{\sigma}{2\pi} - (\pi + O\left(\frac{1}{\sigma \log \sigma}\right)) \tan \frac{\pi \sigma}{2} + O\left(\frac{1}{\sigma}\right)\right] \log \frac{\sigma}{2\pi} = \left(\frac{\pi}{2}\right)^2.
\end{equation}

It follows that $\log \frac{\sigma}{2\pi} \approx \pi \tan \frac{\pi \sigma}{2}$, and $\sigma$ must be close to and to the left of an odd integer.

So we plug $\sigma = 2n + \delta(n)$, ($\delta(n) > 0$) in (33) and solving for $\delta(n)$ we obtain the result. \(\square\)

4. Nonreal zeros of $\zeta''(s)$ in $\sigma < \frac{1}{2}$

Similar to (3) we have
\begin{equation}
\frac{\zeta''}{\zeta''}(s) = \frac{\zeta''(0)}{\zeta''} - 3 - \frac{3}{s - 1} + \sum_{n=1}^{\infty} \left(\frac{1}{s + b_n} - \frac{1}{b_n}\right) + \sum_{\rho_2} \left(\frac{1}{s - \rho_2} + \frac{1}{\rho_2}\right) + \left(\frac{1}{s - b_0} + \frac{1}{b_0} + \frac{1}{s - b_0} + \frac{1}{b_0}\right).
\end{equation}
From Spira [7] we know that $\beta_2 < 5$ for all $\rho_2$, and analogous to (4) (by using (34) at $s = 10$) we find

$$\sum_{\rho_2} \frac{1}{\rho_2^2} < 0.12$$

(35)

(from Spira’s list of $\rho_2$ with $|\gamma_2| < 100$ one calculates $\sum \frac{1}{\rho_2^2} > 0.037$).

**Theorem 3.** (unconditional) There is only one pair of nonreal zeros of $\zeta''(s)$ in the left half-plane.

**Proof.** Consider $\Delta_R \arg \frac{\zeta''}{\zeta'(s)}$ where $R$ is as in the proof of Theorem 2, but with $\sigma_N = -2N - 4$. From our results above, inside $R$ there are $N$ real zeros and two nonreal zeros of $\zeta''$. By Rolle’s Theorem there must be at least $N - 1$ real zeros of $\zeta''$ here. Let $2\kappa$ be the number of nonreal zeros of $\zeta''(s)$. Then

$$\frac{1}{2\pi} \Delta_R \arg \frac{\zeta''}{\zeta'}(s) = Z_3 - Z_2 \geq (N - 1 + 2\kappa) - (N + 2) = 2\kappa - 3.$$ 

We will show that $\Delta_R \arg \frac{\zeta''}{\zeta'}(s) = -2\pi$ in one tour of the rectangle, implying $\kappa \leq 1$. M. Özkul computed that $\zeta''(s)$ has zeros at $-2.11011 \pm i \cdot 2.5842$, so $\kappa = 1$. This computation was based upon evaluating $\int \frac{\zeta'(iv)}{\zeta''}(s) ds$ around various rectangles. The Euler-Maclaurin formula (27) was used with $N = 10$ and $m = 6$ for the integrand. The line integrations were then done by employing MATHEMATICA.

On the three sides of $R$ in the left half-plane the situation is the same as for $\Re \frac{\zeta''}{\zeta'}$, and there is no need to repeat the arguments in the proof of Theorem 2. On the imaginary axis we have, by (34),

$$\Re \frac{\zeta''}{\zeta'}(it) = \frac{\zeta''}{\zeta'}(0) - 3 + \frac{3}{1 + t^2} + \sum_{n=1}^{\infty} \left( \frac{b_n}{b_n^2 + t^2} - \frac{1}{b_n} \right) + \sum_{\rho_2} \frac{1}{\rho_2^2}$$

(36)

$$+ \sum_{\gamma > 0} \left( \frac{-\beta_2}{\beta_2^2 + (\gamma - t)^2} + \frac{-\beta_2}{\beta_2^2 + (\gamma + t)^2} \right)$$

$$+ \frac{2\Re b_0 - \Re b_0}{|b_0|^2} - \Re b_0 \left( \frac{1}{(\Re b_0)^2 + (t - 3b_0)^2} + \frac{1}{(\Re b_0)^2 + (t + 3b_0)^2} \right)$$
\[
\zeta''(it) = \frac{3t}{1 + t^2} - \sum_{n=1}^{\infty} \frac{t}{b_n^2 + t^2} + \sum_{n=1}^{\infty} \frac{\gamma_2 - t}{\beta_2^2 + (\gamma_2 - t)^2} - \frac{t - \Im b_0}{(\Re b_0)^2 + (t - \Im b_0)^2} - \frac{t + \Im b_0}{(\Re b_0)^2 + (t + \Im b_0)^2}.
\]

In (36), bounding the sum over \(\gamma_2\) trivially by 0, and using (35), the value of \(b_0\) and \(\zeta''(0) = 2.993..\) ([1]), we have

\[
\Re \zeta''(it) < 0.0595 + \frac{3}{1 + t^2} + \sum_{n=1}^{\infty} \frac{b_n}{b_n^2 + t^2} - \frac{1}{b_n} 
- \Re b_0 \left( \frac{1}{(\Re b_0)^2 + (t - \Im b_0)^2} + \frac{1}{(\Re b_0)^2 + (t + \Im b_0)^2} \right).
\]

We see that for \(t \geq \Im b_0\) the right-hand side is a strictly decreasing function of \(t\). So, if we find a value \(t_0 > \Im b_0\) making the right-hand side of (38) negative, then we know that for \(t \geq t_0\), \(\Re \zeta''(it) < 0\). To bound the sums over \(b_n\)'s, using (29) and (30) we take \(\hat{b}_n\) and \(\tilde{b}_n\) for \(1 \leq n \leq 4\) satisfying \(\hat{b}_n < b_n < \tilde{b}_n\) and define

\[
a(t) = \sum_{n=1}^{4} \frac{-t^2}{\hat{b}_n^2 + t^2} + \sum_{n=1}^{6} \frac{t^2}{2n((2n)^2 + t^2)} \\
b(t) = \sum_{n=1}^{4} \frac{-t^2}{b_n^2 + t^2} + \sum_{n=1}^{5} \frac{t^2}{2n((2n)^2 + t^2)} \\
c(t) = \sum_{n=1}^{4} \frac{-t}{\hat{b}_n^2 + t^2} + \sum_{n=1}^{6} \frac{t}{2n((2n)^2 + t^2)} \\
d(t) = \sum_{n=1}^{4} \frac{-t}{b_n^2 + t^2} + \sum_{n=1}^{5} \frac{t}{2n((2n)^2 + t^2)}
\]

Then, similar to (25) and (26) we have

\[
b(t) < \sum_{n=1}^{\infty} \frac{-t^2}{\hat{b}_n^2 + t^2} - \frac{1}{2} \left( \psi(1) - \Re \psi(1 + \frac{it}{2}) \right) < a(t), \quad (39)
\]

\[
d(t) < \sum_{n=1}^{\infty} \frac{-t}{\hat{b}_n^2 + t^2} + \frac{1}{2} \Im \psi(1 + \frac{it}{2}) < c(t). \quad (40)
\]
Now by sheer calculation we find that $t_0 = 5.2$ is admissible, and we only need to consider $0 \leq t \leq 5.2$ to determine $\Delta_R \arg \frac{\zeta'''}{\zeta'}(s)$. The quadrants where $\frac{\zeta'''}{\zeta'}(it)$ lies for various $t$’s can be found from the foregoing expressions (e.g. $\frac{\zeta'''}{\zeta'}(i3b_0)$ is in the first quadrant). Also note that as $t \to 0^+$, $\frac{\zeta'''}{\zeta'}(it) \to \frac{\zeta'''}{\zeta'}(0)$ from the first quadrant.

When using (36) to obtain a lower bound for $\Re \frac{\zeta'''}{\zeta'}(it)$ observe that

$$-\frac{\beta_2}{\beta_2^2 + (\gamma_2 - t)^2} \geq -\frac{2\beta_2}{\beta_2^2 + \gamma_2^2}, \quad (0 \leq t \leq \min |\gamma_2|(1 - \frac{1}{\sqrt{\gamma_2}})), $$

and since ([7]) the least $|\gamma_2|$ is 23.27..., for $t \leq 6.8$ the sum over $\gamma_2$ in (36) is $>-\frac{3}{2} \sum_{\rho_2} \frac{1}{\rho_2} > -0.18$. The sum over $\rho_2$ in (37) is equal to

$$2t \sum_{\gamma_2 > 0} \frac{\gamma_2^2 - \beta_2^2 - t^2}{\beta_2^2 + (\gamma_2 - t)^2} \frac{1}{\beta_2^2 + (\gamma_2 + t)^2}, \quad (41)$$

all of the terms in this sum being positive for

$$0 < t < \sqrt{(\min \gamma_2)^2 - (\max \beta_2)^2},$$

i.e. certainly for $0 < t < 22.7$. A trivial lower bound for (41) is 0, and one can do better by including the terms corresponding to known values of $\rho_2$. When using (37) to obtain an upper bound for $\Im \frac{\zeta'''}{\zeta'}(it)$, the sum over $\rho_2$ presents some difficulty. The quantity in (41) is less than

$$2t \sum_{\gamma_2 > 0} \frac{\gamma_2^2 - t^2}{(\gamma_2 - t)^2(\gamma_2 + t)^2} = 2t \sum_{\gamma_2 > 0} \frac{1}{\gamma_2^2 - t^2} < 2.12t \sum_{\gamma_2 > 0} \frac{1}{\gamma_2^2}$$

where the last inequality holds for $0 \leq t \leq 5.2$. However we do not know the value of the last sum. If we cheat and assume RH to the effect that $\beta_2 \geq \frac{1}{2}$, then for $0 < t \leq 5.2$ we have

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\[
\sum_{\gamma_2 > 0} \frac{2t(\gamma_2^2 - \beta_2^2 - t^2)}{[\beta_2^2 + (\gamma_2 - t)^2][\beta_2^2 + (\gamma_2 + t)^2]} < 2t \sum_{\gamma_2 > 0} \frac{1}{\beta_2^2 + (\gamma_2 - t)^2} \\
\leq 2t \sum_{\gamma_2 > 0} \frac{2\beta_2}{\beta_2^2 + \gamma_2^2} \frac{\beta_2^2 + \gamma_2^2}{\beta_2^2 + (\gamma_2 - t)^2} \\
< 2t \left( \frac{\max \beta_2}{\min \gamma_2^2} \right)^2 \sum_{\rho_2} \frac{1}{\rho_2^2},
\]

and we may use (35) to get an upper bound. We do not assume RH, and instead appeal to the graph of \( \frac{\zeta''}{\zeta}(it) \). We see that as \( t \) moves up on the imaginary axis on our contour \( \Delta \arg \frac{\zeta''}{\zeta} \) is roughly \(-2\pi\). This completes the proof. \( \square \)

Next consider the values of \( \mathfrak{R} \frac{\zeta''}{\zeta}(s) \), in the region \( 0 \leq \sigma < \frac{1}{2}, |t| > T \). If we assume RH, then by Theorem 1, \( \mathfrak{R} \frac{1}{s - \rho_2} < 0 \) for all \( \rho_2 \). So from (34), when \( T \geq |\Re b_0| + |\Im b_0| \), (bounding the sums over \( b_n \) as in (39))

\[
\mathfrak{R} \frac{\zeta''}{\zeta}(s) < \frac{\zeta''(0)}{\zeta''} - 3 + \frac{3}{1 + T^2} + \sum_{\rho_2} \frac{1}{\rho_2^2} + g(T) + \frac{2\Re b_0}{|b_0|^2} + \\
+ \left( \frac{1}{2} - \Re b_0 \right) \left( \frac{1}{(T - \Im b_0)^2} + \frac{1}{(T + \Im b_0)^2} \right) + \\
+ \frac{2T^2}{1 + 4T^2} \left( \psi(1) - \Re \psi \left( \frac{5}{4} + \frac{iT}{2} \right) + \frac{1}{4T} \Im \psi \left( \frac{5}{4} + \frac{iT}{2} \right) \right),
\]

where

\[
g(t) = \sum_{n=1}^{4} b_n \left[ \frac{-t^2}{(\frac{t}{2} + b_n)^2 + t^2} \right] + \sum_{n=1}^{4} \frac{t^2}{(2n + 4)[(2n + \frac{9}{2})^2 + t^2]} \\
+ \frac{t^2}{8[(\frac{t}{2})^2 + t^2]} + \frac{t^2}{16[(\frac{t}{2})^2 + t^2]}.
\]

Thus it is seen that \( \mathfrak{R} \frac{\zeta''}{\zeta}(s) < 0 \) in \( 0 \leq \sigma < \frac{1}{2}, |t| \geq 10 \). The region with \( |t| < 10 \) may be swept by integrating \( \frac{\zeta''(s)}{\zeta'(s)} \) around the rectangle, employing the Euler-Maclaurin
formulae (27) for the integrand. This computation was carried out by H.E. Yldirm and no zeros of \( \zeta'''(s) \) were found. Hence we have

**Theorem 4.** The Riemann Hypothesis implies that \( \zeta'''(s) \) has no zeros in the strip \( 0 \leq \sigma < \frac{1}{2} \).

Armed with the methods and results of this paper one may proceed to the investigation of \( \zeta^{(iv)}(s) \).

**Graphs**

The graphs of \( \frac{\zeta^{(k+1)}(it)}{\zeta(it)} \), \( k = 0, 1, 2, 3 \) are plotted below for \( |t| \leq 40 \). The darker parts are for \( -40 \leq t \leq 0 \).
References


[10] D. P. Verma and A. Kaur - Zero-free regions of derivatives of the Riemann zeta function,

2311-2314.

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