Efficient Presentations for Some Direct Products of Groups

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Abstract
In this paper we give efficient presentations for $A_4 \times D_n$, where $n$ is odd number, or $n$ is even number and $(n,3)=1$. We also give efficient presentations for $A_5 \times D_n$ where $n$ is an even or odd number.

1. Introduction
Let $G$ be finite group with a presentation on $n$ generators and $r$ relations. The deficiency of the presentation is $n-r$. A group $H$ of maximal order with the properties that there is a subgroup $A$ with $A \leq Z(H) \cap H'$ and $H/A \cong G$ is called a covering group of $G$. In general, $H$ is not unique but $A$ is unique and is called the Schur multiplier $M(G)$ of $G$. For details see [1, 7, 15].

Schur [9] showed that any presentation for $G$ with $n$ generators requires at least $n+\text{rank}(M(G))$ relations. If $G$ has a presentation with $n$ generators and precisely $n+\text{rank}(M(G))$ relations we say that $G$ is efficient. Not all groups are efficient and examples of soluble groups with trivial multipliers which are not efficient were given by Swan [10] and inefficient groups have been found by Wotherspoon [16]. Further details of such groups are given in [1], [14], [15].

For the finite field $GF(p)$, for a prime $p$, let $SL(2, p)$ denote the group of $2 \times 2$ matrices of determinant 1 over the field $GF(p)$. Define $PSL(2, p) = SL(2, p)/\{ \pm I \}$, where $I$ is the $2 \times 2$ identity matrix.

For any group $G$ we shall use $G'$ and $Z(G)$ to denote the derived group of $G$ and the center of $G$, respectively. We also use the notation $A_4$ and $A_5$ to denote, respectively, the alternating groups of degree four and five. Let $D_n$ denote the dihedral group of order $2n$. 

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Questions concerning the efficiency of direct products have been of considerable interest for a number of years. The first questions concerning the efficiency of direct products were posed by Wiegold in [15]. In particular his questions were whether $PSL(2,5) \times PSL(2,5)$ and $SL(2,5) \times SL(2,5)$ are efficient.

The Schur-Künneth formula [6] gives the Schur multiplier of a direct product:

$$M(G \times H) = M(G) \times M(H) \times (G \otimes H).$$

The first of these questions was answered by Kenne in [8]. He showed that $PSL(2,5)$ is efficient. The second question was answered by Campbell et al. [2]. In [4] C. M. Campbell, E. F. Robertson and P. D. Williams have obtained efficient presentations for certain direct products involving field of the same characteristic. Some work on direct product of groups $PSL(2,p^n)$ for a fixed prime $p$ and different $n$ is done and also some efficient presentations for $PSL(2,q_1) \times PSL(2,q_2)$, $q_1, q_2$ prime power, are given by Vatansever in [11]. In [12] the efficiency of the group $PSL(2,Z_n) \times PSL(2,Z_m)$, for certain $n, m$ is given. In [5] D. M. Gill has obtained efficient presentations of direct products of familiar groups. In [13] the efficiency of the group $PSL(2,7) \times PSL(2,3^2)$ is given.

In this paper we consider the problem of giving efficient presentations for the direct products $A_4 \times D_n$ and $A_5 \times D_n$.

From the Schur-Künneth formula we have:

i) If $n$ is odd then the rank of $M(A_k \times D_n)$ is 1 where $k = 4, 5$.

ii) If $n$ is even then the rank of $M(A_k \times D_n)$ is 2 where $k = 4, 5$.

2. The direct products $A_4 \times D_n$ and $A_5 \times D_n$.

Theorem 1. When $n$ is odd, a presentation of $(2,3,2r+1) \times D_n$ is

$$\langle x, y \mid (xy)^{2r+1}, x^{n+1}y^3, x^2yx^2y^5, y^3 = xy^3x \rangle.$$

Proof. Take $(2,3,2r+1)$ as $\langle a, b \mid a^2, b^3, (ab)^{2r+1} \rangle$ and $D_n$ as $\langle c, d \mid c^2, d^n, (cd)^2 \rangle$.

Define $x = ad$, and $y = bc$, so $x^n = a$, $x^{n+1} = d$, $y^3 = c$, and $y^{-2} = b$. Therefore the direct product of these two groups has presentation

$$\langle x, y \mid x^{2n}, y^6, (x^n y^{-2})^{2r+1}, (y^3 x^{n+1})^2, x^ny^3 = y^3x^n, x^2y^2 = y^2x^2 \rangle.$$
Firstly $(y^3x^{n+1})^2 = y^5x^{n+1}y^{x^{n+1}}$ (using $x^2y^2 = y^2x^2$), for which we have

\[ y = x^{n+1}y^{x^{n+1}} \quad \text{(using } y^6 = 1) \]

and

\[ y = x^{2n+2}y^{2x^{n+2}} = x^2y^2 \quad \text{(using } x^{2n} = 1). \]

So add $y = x^2y^2$ and note that $x^2y^2 = y^2x^2$ is then redundant.

Also,

\[ (y^3x^{n+1})^2 = y^3xy^3x^{2n+1} \quad \text{(using } x^n = y^3x) \]

we have

\[ y^3 = x^3y. \quad \text{(using } x^2n = 1, y^6 = 1) \]

So replace $(y^3x^{n+1})^2 = 1$ by $y^3 = xy^3x$. Then replace $(x^n y^{-2})^{2r+1} = 1$ by $(x^n y^{2r+1})^y = 1$. (Therefore $x^n y^3 = y^3x$ is redundant).

So we now have:

\[ (x, y | x^{2n}, y^6, (x^n y)^{2r+1}y^3, y = x^2y^2, y^3 = xy^3x). \]

But $1 = (x^n y)^{2r+1}y^3 = (x^n y)^{2r+1}y^3 = y^3xy^3$ (using $y = x^2y^2$)

\[ = xyxyxy \quad \text{(using } x^2y^2) \]

\[ = (xy)^2xy^{x^{-1}y}y^3 \quad \text{(using } x^2 = xy^2) \]

\[ = (xy)^{2r+1}xy^{x^{-1}y}y^3 \quad \text{(using } x^2 = xy^2) \]

Replace $(x^n y)^{2r+1}y^3 = 1$ by this.

But now $x^{2n} = 1$ is redundant as from this new relation we see that

\[ x^{-2n-2} = y^3(xy)^{2r+1}y^3(xy)^{2r+1} \]

\[ = y^3(xy)^{2r+1}y^3(xy)^{2r+1} \quad \text{(using } xy^3 = y^3x^{-1}) \]

\[ = x^{-1}y(xy^{-1}x^{2r+1})y^3(xy)^{2r+1} \]

\[ = x^{-1}y(xy)^r(xy)^{2r+1} \quad \text{(using } x^{-1}y = xy^2) \]

\[ = x^{-2}(xy)^{4r+2} \]

\[ = x^{-2}y^{-3}x^{x^{-1}y^{-1}y^{-3}x^{-1}y^{-1}} \quad \text{(using } (xy)^{2r+1}x^{1+n}y^3 = 1) \]

\[ = x^{-2}y^{-6} = x^{-2} \quad \text{(using } y^3 = xy^3x). \]

Consider

\[ H = (x, y | (xy)^{2r+1}x^{1+n}y^3, x^2y^2y^5, y^3 = xy^3x). \]

Now $(1 = x^2y^2y^2 = x^2y^2x^{-2}y^2)$ we have $x^2 = y^2x^2y^4$.

Similarly $(1 = y^3x^2y^2 = x^{-2}y^4x^2y^2)$ we have $x^2 = y^4x^2y^2$. Therefore $y^2x^2 = y^4x^2y^4 = x^2y^2$ and so $y^6 = 1$. Therefore $H$ is a presentation of the direct product.
Corollary 2. When n is odd, an efficient presentation of $A_4 \times D_n$ is
\[ \langle x, y \mid (xy)^3x^{n+1}, x^2yx^2y^3, y^3 = xy^3x \rangle. \]

Corollary 3. When n is odd, an efficient presentation of $A_5 \times D_n$ is
\[ \langle x, y \mid (xy)^5x^{n+1}, x^2yx^2y^3, y^3 = xy^3x \rangle. \]

Theorem 4. When n is even, a presentation of $(2, 3, 4r + 1) \times D_n$ is
\[ \langle x, y \mid x^4y^2x^3 = y^2, (xy)^{4r}, y^{4r+1}, x^2y^{4r+1} = x^2, (xy)^n x, x^2 \rangle. \]

Proof. Take $(2, 3, 4r + 1)$ as $\langle a, b \mid a^2, b^3, (ab)^{4r+1} \rangle$ and $D_n$ as $\langle c, d \mid c^2, d^n, (cd)^2 \rangle$.
Let $x = bcd$ and $y = abcd$, so $x^3 = cd$, $x^4 = b$, $y^{4r+1} = (cd)^{4r+1} = (cd)cd^{2r}cd^2 = (cd.c.d)^2cd^2 = cd^2$ and $y^{4r+2} = ab$.
Therefore $d = x^{-3}y^{4r+1}$, $a = x^{-3}y^{-4r-1}x^{-3}$ and $a = y^{4r+2}x^{-4}$. Hence a presentation of the direct product is:
\[ \langle x, y \mid (y^{4r+2}x^{-4})^2, x^6, y^{2(4r+1)}, (x^{-3}y^{4r+1})^n, x^2y^{4r+1} = y^{4r+1}x^2, x^3y^2 = y^2x^3 \rangle. \]

The first relation is $1 = (y^{4r+2}x^{-4})^{-2}
= x^{-6}(y^{4r+2}x^{-1})^{-2}$
\[ = (xy)^2 \quad \text{(using } x^3y^2 = y^2x^3) \]
and the fourth is $1 = (x^{-3}y^{4r+1})^{-n}
= (x^{-1}y^{4r+1})^{-n}x^{2n}$
\[ = (y^{4r+1}x^2)^n x^{2n} \quad \text{(using } y^{2(4r+1)} = 1) \]
So we have:
\[ \langle x, y \mid (xy)^2, x^6, y^{2(4r+1)}, (x^{4r+1})^n x, x^2y^{4r+1} = y^{4r+1}x^2, x^3y^2 = y^2x^3 \rangle. \]

Now consider:
\[ \langle x, y \mid x^3y^2x^3 = y^2, (xy)^2, y^{4r+1}, x^2y^{4r+1} = x^2, (xy)^n x, x^2 \rangle. \]
First note that $x^3y^2x^3 = y^2$ implies $x^3y^4 = y^4x^3$.
Now for $(xy)^2 = 1$ we have $xy^4x = y^{-4r}$ so
\[ x^3y^{4r}x^3 = x^3y^{-4r}x^2 \]
we have $x^6y^{4r} = x^2y^{-4r}x^2$. (using $x^3y^4 = y^4x^3$)
This is $x^6 = x^2y^{−4r}x^2y^{−4r}$
\[= x^2y^2y^{−4r−1}x^2y^{−4r−1}y\]
\[= x^2yx^2y.\] (using $y^{4r+1}x^2y^{4r+1} = x^2$).

Also, $x^4y^r x^4 = x^3y^{−4r}x^3 = x^6y^{−4r}$, but
\[x^4y^{4r} x^4 = x^4y^{−1}y^{4r+1}x^4\]
\[= x^4y^{−1}x^4y^{4r+1}.\] (using $y^{4r+1}x^2y^{4r+1} = x^2$).

Equating these gives
\[yx^2 = x^4y^{8r+1}.\] (0.1)

We know $x^6 = x^2yx^2y$ so $x^6 = x^2 x^4 y^{8r+1} y$, i.e. $y^{8r+2} = 1$ and hence $y^{4r+1} x^2 y^{4r+1} = x^2$ shows that $x^2 y^{4r+1} = y^{4r+1} x^2$.

As $\gcd(4, 8r + 2) = 2$, $x^3 y^4 = y^4 x^3$ we have $x^3 y^2 = y^2 x^3$ and therefore from $x^3 y^2 x^3 = y^2$, we see that $x^6 = 1$ and we have a presentation of the direct product.
\[\square\]

**Corollary 5.** An efficient presentation of $A_5 \times D_n$, where $n$ is even, is
\[\langle x, y | x^3 y^2 x^3 = y^2, (xy^4)^2, y^5 x^2 y^5 = x^2, (xy^5)^n x^2 n \rangle.\]

**Theorem 6.** An efficient presentation of $A_4 \times D_n$, when $n$ is even and $(n, 3) = 1$, is
\[\langle x, y | x^6, y^{6n−1} = x^3 y x^3, (xy)^2, x = y^3 x y^3 \rangle\]
where $\varepsilon \equiv \frac{n}{2} (\text{mod} 3)$ and $\varepsilon \in \{-1, 1\}$.

**Proof.** $M(A_4 \times D_n) = C_2 \times C_2$. Let $D_n = \langle a, b | a^2, b^n, (ab)^2 \rangle$ and $A_4 = \langle c, d | c^3, d^3, (cd)^2 \rangle$. First take $n = 6r + 2$. Let $x = ac$ and $y = bd$, hence $x^3 = a$, $x^{-2} = c$, $y^{-n} = d$ and using $(n = 6r + 2$ and $(n, 3) = 1$) we obtain $y^{n+1} = b$. So the direct product is
\[\langle x, y | x^6, y^{6n}, y^n = (x^3 y)^2, y^{n−2} = (x^2 y)^2, x^2 y^n = y^n x^3, x^2 y^3 = y^3 x^2 \rangle.\]

Note that $x^3 y^n = y^n x^3$ is redundant. (using $y^n = (x^3 y)^2$)

Also we see that $1 = x^2 y^{−1} y^2 x^{−1} y^{−2}$
\[= x^2 y^{−1} x^2 y^2 y^{−n} \]
\[= x^2 y^{−1} x^2 y (x^3 y)^{−2} \] (using $y^n = (x^3 y)^2$)
we have $(xy)^2 = 1$.

Therefore replace $y^{n−2} = (x^2 y^{−1})^2$ by $(xy)^2 = 1$. 

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Now \((y^{n-1})^3 = (x^3yx^3)^3\) (using \(y^n = (x^3y)^2\))
\[= x^3y^3x^3\] (using \(x^6 = 1\))
we have \(x^3y^3x^3 = 1\) (using \(y^{3n} = 1\))
we obtain \(x = y^3xy^3\). (using \(x^2y^3 = y^3x^2, x^6 = 1\)).

Add the relation \(x = y^3xy^3\). However \(x = y^3xy^3\) we have \(x^2y^3 = y^3x^2\), so the later is redundant. So we see that the direct product can be presented by
\[
\langle x, y \mid x^6, y^{3n}, y^n = (x^3y)^2, (xy)^2, x = y^3xy^3 \rangle.
\]

Consider
\[
\langle x, y \mid x^6, y^{n-1} = x^3yx^3, (xy)^2, x = y^3xy^3 \rangle.
\]
As we just mentioned, \(x = y^3xy^3\) we have \(x^2y^3 = y^3x^2\).
And \((y^{n-1})^3 = (x^3yx^3)^3\)
\[= x^3y^3x^3\] (using \(x^6 = 1\))
Hence
\[y^{3n} = x^3y^3x^3y^3.\] (0.2)

But, \(x = y^3xy^3\) we have \(x^3 = y^3x^3y^3\) (using \(x^2y^3 = y^3x^2\))
we obtain \(x^3y^3x^3y^3 = 1\). (using \(x^6 = 1\))
Therefore, from (0.2), we see that \(y^{3n} = 1\).

Hence a presentation is one of \(A_4 \times D_n\) when \(n \equiv 2(\text{mod} 6)\) is
\[
\langle x, y \mid x^6, y^{n-1} = x^3yx^3, (xy)^2, x = y^3xy^3 \rangle.
\] (0.3)

When \(n \equiv -2(\text{mod} 6)\), consider \(A_4 \times D_{-n}\). Note that \(-n \equiv 2(\text{mod} 6)\) and so using (0.3) we see that
\[
A_4 \times D_{-n} \cong \langle x, y \mid x^6, y^{(-n)-1} = x^3yx^3, x = y^3xy^3, (xy)^2 \rangle.
\]

However as \(D_n \cong \langle a, b \mid a^2, b^n, (ab)^2 \rangle = \langle a, b \mid a^2, b^{-n}, (ab)^2 \rangle \cong D_{-n}\) we see that \(A_4 \times D_{-n} \cong A_4 \times D_{n}\) hence, when \(n \equiv -2(\text{mod} 6)\),
\[
A_4 \times D_n \cong \langle x, y \mid x^6, y^{n-1} = x^3yx^3, x = y^3xy^3, (xy)^2 \rangle,
\]
and the theorem is proved. \(\square\)
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