On the Asymptotics of Fourier Coefficients for the Potential in Hill’s Equation

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Abstract

We consider Hill’s equation \( y'' + (\lambda - q)y = 0 \) where \( q \in L^1[0, \pi] \). We show that if \( l_n \) — the length of the \( n \)-th instability interval — is of order \( O(n^{-k}) \) then the real Fourier coefficients \( a_n, b_n \) of \( q \) are of the same order for \( k = 1, 2, 3 \), which in turn implies that \( q^{(k-2)} \), the \( (k-2) \)th derivative of \( q \), is absolutely continuous almost everywhere for \( k = 2, 3 \).

1. Introduction

We consider the differential equation

\[
y''(t) + (\lambda - q(t))y(t) = 0
\]

on \([0, \pi]\) where \( \lambda \) is a real parameter, \( q(t) \) is integrable over the interval \([0, \pi]\) which may be extended to the real line by periodicity. We associate three types of boundary conditions with (1) over \([0, \pi]\):

- periodic boundary conditions

\[
y(0) = y(\pi), \quad y'(0) = y'(\pi),
\]

- semi-periodic boundary conditions

\[
y(0) = -y(\pi), \quad y'(0) = -y'(\pi),
\]


- auxiliary boundary conditions

\[ y(\tau) = y(\tau + \pi) = 0, \quad (4) \]

where \( 0 \leq \tau < \pi \).

Let \( \lambda_n (n = 1, 2, \ldots) \) denote the periodic eigenvalues of the problem (1) and (2), and \( \mu_n (n = 1, 2, \ldots) \) the semi-periodic eigenvalues of the problem (1) and (3), and \( \Lambda_n (\tau), (n = 1, 2, \ldots) \) the auxiliary eigenvalues of (1) with condition (4). It is well known that, see for example [4],

\[ \lambda_0 < \mu_0 \leq \lambda_1 \leq \lambda_2 \leq \mu_2 \leq \mu_3 < \ldots \]

We define the instability intervals of (1) as follows:

\[ I_0 = (-\infty, \lambda_0), I_{2m+1} = (\mu_{2m}, \mu_{2m+1}), I_{2m+2} = (\lambda_{2m+1}, \lambda_{2m+2}), \]

and for \( n \geq 1 \) their lengths by \( l_n \). It is shown in [10] that (1) with (4) is equivalent to the Dirichlet problem

\[ y''(t) + (\lambda - q(t + \tau))y(t) = 0, \quad (5) \]
\[ y(0) = y(\pi) = 0. \quad (6) \]

Some asymptotic estimates for the eigenvalues and instability intervals of (1) are provided in [6],[7], respectively. We suppose without loss of generality that

\[ \int_0^\pi q(t)dt = 0 \]

and let \( a_n, b_n \) denote the real Fourier coefficients of \( q \) on \([0, \pi]\), i.e,

\[ a_n = \frac{2}{\pi} \int_0^\pi q(t)\cos(2nt)dt, \quad b_n = \frac{2}{\pi} \int_0^\pi q(t)\sin(2nt)dt. \quad (7) \]

As a result of the needs of modern physics, inverse problems became a hot research area. One of the earliest such problems formulated and solved by Ambarzumian[1]. In 1929, he considered the following question:

\[ y''(t) + (\lambda - q(t))y(t) = 0 \]
and

\[ y'' + \lambda y = 0 \]

subject to the boundary conditions

\[ y'(0) = y' (\pi) = 0 \]

with the same eigenvalues. What can be said about \( q(z) \)? Ambarzumian’s answer was that \( q(z) = 0 \).

Borg[4] considered the general problem of what can be said about \( q(z) \) from a knowledge of spectrum. Similar problems have also been investigated by Hochstadt[10] and Ungar[12]. Now, we state a result proven independently by Hochstadt and Ungar.

**Theorem 1.1**[10] If \( q(z) \) is real and integrable, and if all finite instability intervals vanish then \( q(z) = 0 \) almost everywhere.

In this paper we assume that all finite instability intervals are \( O(n^{-k}) \) and show that \( a_n, b_n = O(n^{-k}) \), from which we deduce that \( q^{(k-2)} \) is absolutely continuous a.e.\((k = 2, 3)\).

Central to our analysis is the following theorem of Hochstadt[10] which involves the auxiliary eigenvalues of (1) considered on the interval \( [0; \pi] \)

\[ y(\tau) = y(\tau + \pi) = 0. \]

**Theorem 1.2**[10] The ranges of \( \Lambda_{2m}(\tau) \) and \( \Lambda_{2m+1}(\tau) \), as functions of \( \tau \) are \( [\mu_{2m}, \mu_{2m+1}] \) and \( [\lambda_{2m+1}, \lambda_{2m+2}] \) respectively.

**Remark:** We make use of the Theorem 1.2 in the sense that if all finite instability intervals are \( O(n^{-k}) \) then \( \Lambda_n(\tau_2) - \Lambda_n(\tau_1) = O(n^{-k}) \) for any \( \tau_1, \tau_2 \in [0, \pi] \).

We also state a sequence of Lemmas which will be used in proving our results. Let

\[ c_n = \frac{1}{\pi} \int_0^\pi q(z)e^{-2inz}dz \tag{8} \]
be the Fourier coefficient of \( q(z) \) over \([0, \pi]\).

**Lemma 1.1** [10] Let \( q(z) \) be periodic with period \( \pi \), integrable over \([0, \pi]\) and such that

\[
c_n = O\left(\frac{1}{n^2}\right)
\]

as \( n \to \infty \). Then \( q(z) \) is absolutely continuous almost everywhere.

**Lemma 1.2** Let \( q(z) \) be periodic with period \( \pi \), integrable over \([0, \pi]\) and such that

\[
c_n = O\left(\frac{1}{n^3}\right)
\]

as \( n \to \infty \). Then \( q'(z) \) is absolutely continuous almost everywhere.

**Proof.** The proof of Lemma 1.1 goes through.

**Lemma 1.3** [9] For \( k = 1, 2, 3, \ldots, \tau \leq t \leq \tau + \pi \)

\[
\Theta(t, \tau) - \Theta_k(t, \tau) = o(\Lambda^{-k/2})
\]

as \( \Lambda \to \infty \).

**Lemma 1.4** [2] For \( q \) integrable and for any \( x_1, x_2 \) such that \( \tau \leq x_1 < x_2 \leq \tau + \pi \)

\[
\int_{x_1}^{x_2} q(t) \sin(2\Lambda^{1/2}t)dt = o(1)
\]

as \( \Lambda \to \infty \).

Now, we introduce the function \( \Theta(t, \Lambda, \tau) \), the so-called modified Prüfer transformation of [2], which is defined for any given solution of (1) as

\[
tan\Theta(t, \Lambda, \tau) = \frac{\Lambda^{1/2}y(t, \tau)}{y'(t, \tau)},
\]
for $\tau \leq t \leq \tau + \pi$. This fixes $\Theta$ to within additive multiples of $\pi$. For definiteness we assume that $0 \leq \Theta(t, \tau) \leq \pi$ and observe that the boundary conditions (4) correspond to

$$\Theta(t, \tau) = 0, \quad \Theta(t, \tau + \pi) = (n + 1)\pi,$$  \hspace{1cm} (9)

similarly the boundary conditions (6) correspond to

$$\Theta(t, 0) = 0, \quad \Theta(t, \pi) = (n + 1)\pi.$$ \hspace{1cm} (10)

From now on, we suppress the dependence of $\Theta$ on $\Lambda$ and write $\Theta(t, \tau)$ instead of $\Theta(t, \Lambda, \tau)$. Under the Prüfer transformation the differential equation corresponding to (1) can be written as

$$\Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2} \Lambda^{-1/2} q(t) + \frac{1}{2} \Lambda^{-1/2} q(t) \cos(2\Theta(t, \tau)),$$ \hspace{1cm} (11)

and from (11)

$$\Theta(t, \tau) = (\Lambda^{1/2})(t - \tau) - \frac{1}{2} \Lambda^{-1/2} \int_{\tau}^{t} q(s) ds + \frac{1}{2} \Lambda^{-1/2} q(t) \int_{\tau}^{t} q(s) \cos(2\Theta(s, \tau)) ds.$$ \hspace{1cm} (12)

We define a sequence of approximating functions for (12) as follows:

$$\Theta_1(t, \tau) := (\Lambda^{1/2})(t - \tau) - \frac{1}{2} \Lambda^{-1/2} \int_{\tau}^{t} q(s) ds,$$

$$\Theta_{k+1}(t, \tau) := \Theta_1(t, \tau) + \frac{1}{2} \Lambda^{-1/2} \int_{\tau}^{t} q(s) \cos(2\Theta_k(s, \tau)) ds$$ \hspace{1cm} (13)

for $k = 1, 2, \ldots$, and $\tau \leq t \leq \tau + \pi$.

2. The Results

**Theorem 2.1** For any integer $k$, the auxiliary eigenvalues of (1), as functions of $\tau$, satisfy

$$(n + 1)\pi = \Lambda_n^{1/2}(\tau)\pi + \frac{1}{2} \Lambda_n^{-1/2} \int_{0}^{\tau} q(t + \tau) \cos(2\Theta_k(t, \tau)) dt + O(n^{-(k+1)})$$ \hspace{1cm} (14)

as $n \to \infty.$
Proof. We consider the differential equation (5) with the boundary conditions (6). From (11) we get

\[ \Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2} \Lambda^{-1/2} q(t + \tau) + \frac{1}{2} \Lambda^{-1/2} q(t + \tau) \cos(2\Theta(t, \tau)). \]  

(15)

We also know from Lemma 1.3 that

\[ \Theta(t, \tau) - \Theta_k(t, \tau) = o(\Lambda^{-k/2}), \]

(16)

so that

\[ \cos(2\Theta(t, \tau)) = \cos(2\Theta_k(t, \tau)) + O(\Lambda^{-k/2}). \]  

(17)

Substituting (17) into (15) we obtain

\[ \Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2} \Lambda^{-1/2} q(t + \tau) + \frac{1}{2} \Lambda^{-1/2} q(t + \tau) \cos(2\Theta_k(t, \tau)) + O(\Lambda^{-k+1/2}). \]  

(18)

Integrating (18) with respect to \( \tau \) on \([0, \pi]\) and using (10) we complete the proof.

Corollary 2.1 As \( n \to \infty \) the auxiliary eigenvalues of (1), as functions of \( \tau \), satisfy

\[ \Lambda_n^{1/2}(\tau) = (n + 1) + \frac{1}{(n + 1)} F_1(n, \tau) + O(n^{-2}) \]  

(19)

where

\[ F_1(n, \tau) = -\frac{1}{2\pi} \int_0^\pi q(t + \tau) \cos(2(n + 1)t) dt. \]  

(20)

Proof. From (13), we see that

\[ \cos(2\hat{\eta}_1(t, \tau)) = \cos(2\Lambda^{1/2}t) + O(\Lambda^{-1/2}). \]  

(21)

Substituting (21) into (14) and using reversion we complete the proof.

Theorem 2.2 Let \( q(t) \) be real-valued integrable function on \([0, \pi]\). If \( l_n = O(n^{-1}) \), then \( a_n, b_n = O(n^{-1}) \) as \( n \to \infty \).
Proof. It is easily seen that

\[
F_1(n, \tau) = -\frac{1}{4} \{\cos(2(n+1)\tau)a_{n+1} + \sin(2(n+1)\tau)b_{n+1}\},
\]

(22)

where \(a_{n+1}\) and \(b_{n+1}\) are defined in (7), and \(F_1(n, \tau)\) is given by (20). From Corollary 2.1, for any \(\tau_1, \tau_2 \in [0, \pi]\)

\[
\Lambda_n^{1/2}(\tau_2) - \Lambda_n^{1/2}(\tau_1) = \frac{1}{4(n+1)} \{[\cos(2(n+1)\tau_1) - \cos(2(n+1)\tau_2)] a_{n+1}
+ [\sin(2(n+1)\tau_1) - \sin(2(n+1)\tau_2)] b_{n+1}\} + O(n^{-2})
= \frac{1}{2(n+1)} \{\sin((n+1)(\tau_1 + \tau_2)) \sin((n+1)(\tau_2 - \tau_1)) a_{n+1}
+ \cos((n+1)(\tau_1 + \tau_2))\sin((n+1)(\tau_1 - \tau_2)) b_{n+1}\} + O(n^{-2})
= \frac{1}{2(n+1)} \sin((n+1)(\tau_2 - \tau_1)) \{\sin((n+1)(\tau_1 + \tau_2)) a_{n+1}
- \cos((n+1)(\tau_1 + \tau_2)) b_{n+1}\} + O(n^{-2}).
\]

(23)

On the other hand, from the assumption that \(l_n = O(n^{-1})\) and Lemma 1.4.

\[
\Lambda_n(\tau_2) - \Lambda_n(\tau_1) = O(n^{-1})
\]

and hence of

\[
\Lambda_n^{1/2}(\tau_2) - \Lambda_n^{1/2}(\tau_1) = \frac{\Lambda_n(\tau_2) - \Lambda_n(\tau_1)}{\Lambda_n^{1/2}(\tau_2) + \Lambda_n^{1/2}(\tau_1)} = O(n^{-2}).
\]

(24)

The result follows from (23) and (24).

Corollary 2.2 As \(n \to \infty\) the auxiliary eigenvalues of (1), as functions of \(\tau\), satisfy

\[
\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)} F_1(n, \tau) + \frac{1}{(n+1)^2} F_2(n, \tau) + O(n^{-3}),
\]

(25)

where \(F_1(n, \tau)\) is given by (20) and

\[
F_2(n, \tau) = -\frac{1}{2\pi} \int_0^\pi q(t + \tau) \left( \int_0^t q(s + \tau) ds \right) \sin(2(n+1)t) dt.
\]

21
\[ + \frac{1}{2\pi} \int_0^\pi q(t + \tau) \left( \int_0^t q(s + \tau) \cos(2(n + 1)s) ds \right) \sin(2(n + 1)t) dt. \] (26)

**Proof.** From (13) we observe that for \( k = 1 \)

\[ \cos(2\Theta_2(t, \tau)) = \cos(2\Lambda^{1/2}t) + \Lambda^{-1/2} \sin(2\Lambda^{1/2}t) \left( \int_0^t q(s + \tau) ds \right) \]

\[ - \Lambda^{-1/2} \sin(2\Lambda^{1/2}t) \left( \int_0^t q(s + \tau) \cos(2\Lambda^{1/2}s) ds \right) + O(\Lambda^{-1}). \] (27)

Substituting (27) into (14) and using reversion we complete the proof.

**Theorem 2.3** Let \( q(t) \) be real-valued integrable function on \([0, \pi] \). If \( l_n = O(n^{-2}) \) then \( a_n, b_n = O(n^{-2}) \) as \( n \to \infty \).

**Proof.** Since \( l_n = O(n^{-2}) \) by assumption, it is \( O(n^{-1}) \) as well. Hence \( a_n, b_n = O(n^{-1}) \) by Theorem 2.2. From this and Lemma 1.4 we observe that

\[ F_2(n, \tau) = O(n^{-1}). \]

Therefore, (25) reduces to

\[ \Lambda_n^{1/2}(\tau) = (n + 1) + \frac{1}{(n + 1)} F_1(n, \tau) + O(n^{-3}). \]

By a similar argument in Theorem 2.2, we complete the proof.

**Corollary 2.3** Let \( q(t) \) be a real-valued integrable function on \([0, \pi] \). If \( l_n = O(n^{-2}) \) as \( n \to \infty \) then \( q(t) \) is absolutely continuous almost everywhere.

**Proof.** It follows from Theorem 2.3 and Lemma 1.1

**Corollary 2.4** As \( n \to \infty \) the auxiliary eigenvalues of (1), as functions of \( \tau \), satisfy

\[ \Lambda_n^{1/2}(\tau) = (n + 1) + \frac{1}{(n + 1)} F_1(n, \tau) + \frac{1}{(n + 1)^2} F_2(n, \tau) + \frac{1}{(n + 1)^3} F_3(n, \tau) + O(n^{-4}), \] (28)

where \( F_1(n, \tau) \) is given by (20), \( F_2(n, \tau) \) is given by (26) and
\[
F_3(n, \tau) = -\frac{1}{4(n+1)^3\pi} \int_0^\pi q(t+\tau)\left[\int_0^t q(s+\tau)ds - \int_0^t q(s+\tau)\cos(2(n+1)s)ds\right]^2 \\
\times \cos(2(n+1)t)dt \\
+ \frac{1}{2(n+1)^3\pi} \int_0^\pi q(t+\tau)\left[\int_0^t q(s+\tau)\left(\int_0^s q(l+\tau)dl\right)\sin(2(n+1)s)\right] \\
\times \sin(2(n+1)t)dt. \tag{29}
\]

**Proof.** From (13) for \(k=2\) we have
\[
\cos(2\Theta_3(t, \tau)) = \cos(2\Lambda^{1/2}t) + \Lambda^{-1/2}\sin(2\Lambda^{1/2}t)\left(\int_0^t q(s+\tau)ds\right) \\
- \Lambda^{-1/2}\sin(2\Lambda^{1/2}t)\left(\int_0^t q(s+\tau)\cos(2\Lambda^{1/2}s)ds\right) \\
- \frac{1}{2}\Lambda^{-1}\left(\int_0^t q(s+\tau)ds - \int_0^t q(s+\tau)\cos(2\Lambda^{1/2}s)ds\right)^2 \cos(2\Lambda^{1/2}t) \\
- \Lambda^{-1}\left(\int_0^t q(s+\tau)\left(\int_0^s q(l+\tau)dl\right)\sin(2\Lambda^{1/2}s)\right)\sin(2\Lambda^{1/2}t) \\
+ O(\Lambda^{-3/2}). \tag{30}
\]
Substituting (30) into (14), and using reversion we complete the proof.

**Theorem 2.4** Let \(q(t)\) be a real-valued integrable function on \([0, \pi]\). If \(l_n = \Theta(n^{-3})\) then \(a_n, b_n = \Theta(n^{-3})\) as \(n \to \infty\).

**Proof.** Since \(l_n = \Theta(n^{-3})\) by assumption, it is \(O(n^{-2})\) as well. Hence \(a_n, b_n = O(n^{-2})\) by Theorem 2.3. Therefore, the terms involving \(F_2(n, \tau)\) and \(F_3(n, \tau)\) in (28) are included in the error term by Lemma 1.4. By a similar argument in Theorem 2.2 we complete the proof.

**Corollary 2.5** Let \(q(t)\) be a real-valued function on \([0, \pi]\). If \(l_n = \Theta(n^{-3})\) then \(q'(t)\) is absolutely continuous almost everywhere.

**Proof.** It follows from Theorem 2.4 and Lemma 1.2.

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References


