Uniquely strongly clean triangular matrices

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Received: 06.08.2014 • Accepted/Published Online: 31.03.2015 • Printed: ...201

Abstract: A ring \( R \) is uniquely (strongly) clean provided that for any \( a \in R \) there exists a unique idempotent \( e \in R \) \((e \in comm(a))\) such that \( a - e \in U(R) \). We prove, in this note, that a ring \( R \) is uniquely clean and uniquely bleached if and only if \( R \) is abelian, \( T_n(R) \) is uniquely strongly clean for all \( n \geq 1 \), i.e. every \( n \times n \) triangular matrix over \( R \) is uniquely strongly clean, if and only if \( R \) is abelian, and \( T_n(R) \) is uniquely strongly clean for some \( n \geq 1 \). In the commutative case, more explicit results are obtained.

Key words: Uniquely strongly clean ring, uniquely bleached ring, triangular matrix ring

1. Introduction

Throughout this article, all rings are associative with unity. We write \( U(R) \) for the set of all units in \( R \). \( T_n(R) \) stands for the ring of all \( n \times n \) triangular matrices over a ring \( R \). Let \( a, b \in R \). We denote the map from \( R \) to \( R : x \mapsto ax - xb \) by \( l_a - r_b \). We write \( M_n(R) \) for the ring of all \( n \times n \) matrices over the ring \( R \). The commutant of an element \( a \) in a ring \( R \) is defined by \( comm(a) = \{ x \in R \mid xa = ax \} \). \( \mathbb{N} \) is the set of all natural numbers.

A ring \( R \) is strongly clean provided that for any \( a \in R \) there exists an idempotent \( e \in comm(a) \) such that \( a - e \in U(R) \). Strongly clean triangular matrices are extensively studied by many authors, e.g., [1] and [3]. A ring \( R \) is called uniquely clean provided that for any \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in U(R) \). Many characterizations of such rings are studied in [2, 3, 4, 10] and [11]. Following Chen et al. [5], a ring \( R \) is called uniquely strongly clean provided that for any \( a \in R \) there exists a unique idempotent \( e \in comm(a) \) such that \( a - e \in U(R) \). Uniquely strong cleanness behaves very differently from the properties of uniquely clean rings (cf. [5]). In general, matrix rings do not have such properties (see [13, Proposition 11.8]). Thus, it is attractive to investigate uniquely strong cleanness of triangular matrices over a ring. Chen et al. proved that if \( R \) is commutative, then \( R \) is uniquely clean if and only if \( T_n(R) \) is uniquely strongly clean for all \( n \geq 1 \) if and only if \( T_n(R) \) is uniquely strongly clean for some \( n \geq 1 \).

[5, Question 12] and [13, Question 11.13] asked if “commutative” in the preceding result can be replaced by “abelian”. The motivation of this note is to explore this problem. Following [7], a ring \( R \) is uniquely bleached provided that for any \( a \in J(R), b \in U(R), l_a - r_b, \) and \( l_b - r_a \) are isomorphism. We prove, in this note, that \( R \) is uniquely clean and uniquely bleached if and only if \( R \) is abelian, \( T_n(R) \) is uniquely strongly clean for all

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2010 AMS Mathematics Subject Classification: 16U99, 16S99.

1
Theorem 2

Let $U_n(R)$ be a result holds for $(1)$ $l_R = 1$.

Assume that $a_R = 1$.

We get $e_1 = 0$ and $e_{22} = 1$. Thus, $E = \begin{bmatrix} 0 & x \\ x & 1 \end{bmatrix}$ for some $x \in R$. It follows from $EA = AE$ that $ax - xb = r$.

Assume that $ay - yb = r$. Then we have an idempotent $F = \begin{bmatrix} 0 & y \\ y & 1 \end{bmatrix}$ such that $A - F \in U(T_2(R))$ and $AF = FA$. By the uniqueness of $E$, we get $x = y$. Therefore, $L_a - r_b : R \rightarrow R$ is an isomorphism. Likewise, $b - r_a : R \rightarrow R$ is an isomorphism. Accordingly, $R$ is uniquely bleached, as asserted. $\square$

The following theorem is a generalization of Theorem 1 in [6].

Theorem 2 Let $R$ be a ring. Then the following are equivalent:

1. $R$ is uniquely clean and uniquely bleached.

2. $R$ is abelian, and $T_n(R)$ is uniquely strongly clean for all $n \in \mathbb{N}$.

3. $R$ is abelian, $T_n(R)$ is uniquely strongly clean for some $n \in \mathbb{N}$.

Proof (1) $\Rightarrow$ (2) In view of [10, Theorem 20], $R$ is abelian. Clearly, the result holds for $n = 1$. Assume that the result holds for $n(n \geq 1)$. Let $A = \begin{bmatrix} a_{11} & \alpha \\ A_1 \end{bmatrix} \in T_{n+1}(R)$ where $a_{11} \in R$, $\alpha \in M_{1 \times n}(R)$, and $A_1 \in T_n(R)$.

Since $R$ is uniquely clean, we can find a unique idempotent $e_{11} \in R$ such that $u_{11} := a_{11} - e_{11} \in U(R)$ and $a_{11}e_{11} = e_{11}a_{11}$. Furthermore, we have a unique idempotent $E_1 \in T_n(R)$ such that $U_1 := A_1 - E_1 \in U(T_n(R))$ and $A_1E_1 = E_1A_1$; hence, $U_1E_1 = E_1U_1$. Let $E = \begin{bmatrix} e_{11} & x \\ E_1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & \alpha - x \\ U_1 \end{bmatrix}$, where $x \in M_{1 \times n}(R)$. Observing that $E^2 = E \Leftrightarrow e_{11}x + xE_1 = x$; $UE = EU \Leftrightarrow u_{11}x + (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1$, (i) (ii)

and then combining (i) with (ii) yields that $(u_{11} + 2e_{11} - 1)x - xU_1 = e_{11}\alpha - \alpha E_1$. (iv)

It is enough to show that there exists a unique $x \in M_{1 \times n}(R)$ such that $(*)$ holds. In view of [10, Theorem 20], $R/J(R)$ is Boolean, and so $2 \in J(R)$. Furthermore, $u_{11} \in 1 + J(R)$. This shows that $u_{11} + 2e_{11} - 1 \in J(R)$. 2
Write \( x = [ x_1 \; \cdots \; x_n ] \), \( e_{11} \alpha - \alpha E_1 = [ v_1 \; \cdots \; v_n ] \), and \( U_1 = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{22} & \cdots & c_{2n} \\ \vdots \\ c_{nn} \end{bmatrix} \) where each \( c_{ii} \in U(R) \). The equation (*) is equivalent to the \( n \) equations:

\[
(u_{11} + 2e_{11} - 1)x_1 - x_1c_{11} = v_1;
\]

\[
(u_{11} + 2e_{11} - 1)x_2 - x_2c_{22} = v_1 + x_1c_{12};
\]

\[
\vdots
\]

\[
(u_{11} + 2e_{11} - 1)x_n - x_nc_{nn} = v_n + x_1c_{1n} + \cdots + x_{n-1}c_{(n-1)n}.
\]

As \( R \) is uniquely bleached, we have a unique \( x_i \in R \) (\( i = 1, \ldots, n \)), and so there exists a unique \( x \) such that (*) holds. Further, we see that

\[
(u_{11} + 2e_{11} - 1)x(e_{11}I_n + E_1) - x(e_{11}I_n + E_1)U_1 = \alpha(e_{11}I_n - E_1)(e_{11}I_n + E_1)
\]

\[
= \alpha(e_{11}I_n - E_1)
\]

\[
= (u_{11} + 2e_{11} - 1)x - xU_1
\]

because \( E_1U_1 = U_1E_1 \). Set \( y = x(e_{11}I_n + E_1) \). This implies that \((u_{11} + 2e_{11} - 1)(y - x) - (y - x)U_1 = 0\).

Write \( y - x = [ z_1 \; \cdots \; z_n ] \). Then

\[
(u_{11} + 2e_{11} - 1)z_1 - z_1c_{11} = 0;
\]

\[
(u_{11} + 2e_{11} - 1)z_2 - z_2c_{22} = z_1c_{12};
\]

\[
\vdots
\]

\[
(u_{11} + 2e_{11} - 1)z_n - z_nc_{nn} = z_1c_{1n} + \cdots + z_{n-1}c_{(n-1)n}.
\]

Since \( R \) is uniquely bleached, we get each \( z_i = 0 \), and so \( y = x \). This gives that \( e_{11}x + xE_1 = x \) and \( u_{11}x - (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1 \). Therefore, we have a uniquely strongly clean expression

\[
A = \begin{bmatrix} e_{11} & x \\ E_1 \end{bmatrix} + \begin{bmatrix} a_{11} - e_{11} & \alpha - x \\ A_1 - E_1 \end{bmatrix}.
\]

By induction, \( T_n(R) \) is uniquely strongly clean for all \( n \in \mathbb{N} \).

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1) In light of [5, Example 5], both \( R \) and \( T_2(R) \) are uniquely strongly clean, hence the result by [10, Theorem 20] and Lemma 1. \( \square \)

**Remark 3** Examples of uniquely bleached rings include the ring with nil Jacobson radical, the ring for which some power of each element in \( J(R) \) is central, and commutative rings.

The double commutant of an element \( a \) in a ring \( R \) is defined by \( comm^2(a) = \{ x \in R \mid xy = yx \text{ for all } y \in comm(a) \} \). Clearly, \( comm^2(a) \subseteq comm(a) \). This concept is closely related to quasipolar, perfectly clean, and pseudopolar elements (for details see \[4, 8, 9, 12\]). We end this note by a more explicit result than [5, Theorem 10].

**Theorem 4** Let \( R \) be a commutative ring, and let \( n \in \mathbb{N} \). Then the following are equivalent:

(1) \( R \) is uniquely clean.

(2) For any \( A \in T_n(R) \), there exists a unique idempotent \( E \in comm^2(A) \) such that \( A - E \in U(T_n(R)) \).
Proof. (1) $\Rightarrow$ (2) For any $A \in T_n(R)$, we claim that there exists an idempotent $E \in \text{comm}^2(A)$ such that $A - E \in U(T_n(R))$. Suppose that the result holds for $n - 1, n \geq 2$. Let $A = \begin{bmatrix} a_{11} & \alpha \\ \alpha & A_1 \end{bmatrix} \in T_n(R)$, where $a_{11} \in R, \alpha \in M_{1 \times (n-1)}(R)$ and $A_1 \in T_{n-1}(R)$. Since $R$ is a commutative uniquely clean ring, there exists a unique idempotent $E = \begin{bmatrix} e_{11} & x \\ \ast \ast \ast \end{bmatrix}$ such that $A - E \in U(T_n(R))$ and $E \in \text{comm}(A)$ by Theorem 2. Write $A - E = \begin{bmatrix} u_{11} & \alpha - x \\ u_{11} & \ast \ast \ast \end{bmatrix}$. According to the proof of Theorem 2, we know that $\alpha(E_1 - e_{11}I_{n-1}) = x(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1})$ by (*) and $A_1$ is uniquely strongly clean with $E_1$. This implies that $E_1 \in \text{comm}^2(A_1)$ by induction. For any $X = \begin{bmatrix} x_{11} & \beta \\ \ast \ast \ast \end{bmatrix} \in \text{comm}(A)$, we have $x_{11}\alpha + \beta A_1 = a_{11}\beta + aX_1$, and so $\alpha(X_1 - x_{11}I_{n-1}) = \beta(A_1 - a_{11}I_{n-1})$. We check that

$$
\beta(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) = \alpha(X_1 - x_{11}I_{n-1})(E_1 - e_{11}I_{n-1})
$$

because $E_1 \in \text{comm}^2(A_1)$ and $X_1 \in \text{comm}(A_1)$. Moreover,

$$
\beta(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) = \beta(E_1 - e_{11}I_{n-1})(U_1 - (e_{11} + u_{11})I_{n-1})
$$

This shows that $\beta(E_1 - e_{11}I_{n-1}) = x(X_1 - x_{11}I_{n-1})$ since $U_1 - (u_{11} + 2e_{11} - 1)I_{n-1} \in U(T_{n-1}(R))$. Thus, we get $e_{11}\beta + xX_1 = x_{11}x + \beta E_1$; hence, $EX =XE$. That is, $E \in \text{comm}^2(A)$, as claimed.

(2) $\Rightarrow$ (1) Let $a \in R$. Then $A = \text{diag}(a, a, \ldots, a) \in T_n(R)$. Hence, we can find a unique idempotent $E = [e_{ij}] \in \text{comm}^2(A)$ such that $A - E \in U(T_n(R))$. This implies that $e_{11} \in R$ is an idempotent and $a - e_{11} \in U(R)$. Suppose that $a - e \in U(R)$ with an idempotent $e \in R$. Then $F = \text{diag}(e, e, \ldots, e) \in T_n(R)$ is an idempotent. Further, $F \in \text{comm}^2(A)$, and that $A - F \in U(T_n(R))$. By the uniqueness, we get $E = F$, and then $e = e_{11}$. Therefore $R$ is uniquely clean, as asserted. \hfill $\Box$

The next result showed that if $R$ is commutative uniquely clean, then both $A$ and $-A$ are uniquely strongly clean for any $A \in T_n(R)$.

**Corollary 5** Let $R$ be a commutative uniquely clean ring. Then for any $A \in T_n(R)$, there exists a unique idempotent $E \in \text{comm}(A)$ such that $A - E, A + E \in U(T_n(R))$.

**Proof** In view of Theorem 4, we have a unique idempotent $E \in \text{comm}^2(A^2)$ such that $A^2 - E$ is invertible. Obviously, $EA = AE$. Then $A - E$ and $A + E$ are invertible. On the other hand, if $A - F$ and $A + F$ are invertible for some idempotent $F$ that commutes with $A$, then $A^2 - F$ is invertible. Then $A - E, A - F \in U(T_n(R))$ and $EA = AE, FA = AF$. By Theorem 2, we have $E = F$, as desired. \hfill $\Box$
Acknowledgments
The authors are grateful to the referee for his/her helpful suggestions, which led to the new version being clearer. This research was supported by the Natural Science Foundation of Zhejiang Province (LY13A010019) and the Scientific and Technological Research Council of Turkey (TÜBİTAK-2221 Visiting Scientists Fellowship Programme).

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