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Ellipsoid based $\mathcal{L}_2$ controller design for LPV systems with Saturating actuators

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Abstract

This paper addresses the $\mathcal{L}_2$ gain control problem for disturbance attenuation in Linear Parameter Varying (LPV) Systems having saturating actuators when the system is subjected to $\mathcal{L}_2$ disturbances. In the presented method, saturating actuator is expressed analytically with a convex hull of some linear feedback which let us construct $\mathcal{L}_2$ control problem via Linear Matrix Inequalities (LMIs) which are obtained by some ellipsoids. It is shown that the stability and disturbance rejection capabilities of the control system are all measured by means of these nested ellipsoids where the inner ellipsoid covers the initial conditions for states whereas the outer ellipsoid designates the $\mathcal{L}_2$ gain of the system. It is shown that the performance of the controller is highly related by the topology of these ellipsoids. Finally, the efficiency of the proposed method is successfully demonstrated through simulation studies on a single-track vehicle dynamics having some linear time-varying parameters.

Key Words: *LMI, linear parameter varying (LPV) systems, single-track vehicle model*

1. Introduction

Disturbance rejection problem in linear systems with saturating actuators have recently started to attract attention. It is due to the fact that actuator saturation could defects controller performance and even more could lead the system to instability. The studies on actuator saturation problem in literature can be grouped in two categories depending on how the disturbances act on the system. First group includes the ones which are input-additive [1], [2], while the second group analyzes the ones which can not be classified as input-additive[3], [4],[5], [6], [7]. It is shown and proven in literature that in case of systems of which disturbances affect input additively, strong stability and very good performance could be obtained. But it is difficult to claim the same for the later case. On the other hand, if the disturbances acting on the system have bounded magnitude, the stability of system can only be analyzed via invariant ellipsoids. As in the case of this study, if there is no

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restrictions on initial conditions and disturbances except energy bound, solution can only be obtained through nested ellipsoids [6], [3].

In this work, the saturation is guaranteed to be in a group of convex linear feedback hull of control signal as described in [5], [6]. The idea behind is on nested ellipsoids. Another effort in literature uses modified sector conditions for actuator saturation [3], [7]. The goal of this work is to provide a method to enlarge the set containing the initial conditions for states and to present multi-objective optimization method to construct a state-feedback Linear Parameter Varying (LPV) controller for the minimization of closed-loop $\mathcal{L}_2$ gain for LPV systems. To the best of authors knowledge, there does not exist a paper in literature that deals with the $\mathcal{L}_2$ gain LPV control of actuator saturated systems. The applicability of the proposed method is also demonstrated on a single track unstable model for a vehicle at the end.

The rest of the paper is organized as follows: The problem is formulated in Section 2. Mathematical background is constructed for main results in Section 3. Main outcomes of the study is demonstrated, in Section 4. The proposed method is extended to the LPV systems in Section 5. In Section 6, the efficiency of proposed methodology is demonstrated on the single-track vehicle model both for the LPV case and nominal case. Finally, in Section 7, results are discussed and possible future studies are addressed.

2. Problem definition

Let us consider,

$$\begin{aligned} \dot{x} &= Ax + B_1\text{sat}(u) + B_2 w \\ z &= Cx, \qquad x(0) = x_0 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ real-time measurable states, $x(0) = x_0$ initial conditions which are generally not needed to be identical to zero, $u \in \mathbb{R}^m$ denotes the control inputs, $w \in \mathbb{R}^q$ are the disturbances acting on the system. $\text{sat}(\cdot)$ is the standard saturation function with unity saturation level, *i.e.*, $\text{sat}(u) = \begin{bmatrix}\text{sat}(u_1) & \text{sat}(u_2) & \cdots & \text{sat}(u_m)\end{bmatrix}^{\mathrm{T}}$, where $\text{sat}(u_i) = \text{sign}(u_i)\min\{1, |u_i|\}$. Please note that here we mildly harm the notation by using $\text{sat}(\cdot)$ to denote both the scalar valued and vector valued saturation functions. Also note that any non-unity input level can be easily converted to a unity one by simply scaling B_1 matrix appropriately. $z \in \mathbb{R}^p$ is performance or the controlled output. Then A, B_1, B_2 and C are system matrices of appropriate dimensions. Let us assume that disturbances acting on the system penetrate from the set of $\mathcal{W}$ having bounded energy. Assuming $1/\delta \in \mathbb{R}_+$ is known upper energy bound for disturbances, then one can define this set as,

$$\mathcal{W} := \left\{ w : \mathbb{R}_+ \to \mathbb{R}^q \ : \ \int_0^\infty w^{\mathrm{T}}(t)w(t)dt \leq \frac{1}{\delta} \right\}. \tag{2}$$

This study aims to ensure the stability of the closed-loop system with a finite $\mathcal{L}_2$ gain (stability problem), construct the largest set of initial conditions that closed-loop system can tolerate with ensuring the closed-loop stability and construct the largest set of initial conditions that closed-loop system can tolerate with ensuring the closed-loop stability with a control law of $u(\rho(t), t) = F(\rho(t))x(t)$ when all initial conditions are different from zero ($x_0 \neq 0$):

3. Mathematical background and notation

For given positive definite matrix, $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and positive number, r, ellipsoid, $\mathcal{E}$ is defined as, $\mathcal{E}(P,r) := \left\{x \in \mathbb{R}^n \,:\, x^{\mathrm{T}}Px \le r\right\}$. On the other hand, we define the set of states for which saturation does not occur as, $\mathcal{L}(D) := \{x \in \mathbb{R}^n \,:\, |D_i x| \le 1 \; i = 1,\dots,m\}$, where, D_i represents i^{th} row of gain matrix, D.

In this paper, to be able to reach the control objectives listed in the previous section, nonlinear saturation function, $\mathrm{sat}(\cdot)$, is forced to be defined in a convex hull of linear feedbacks. Thanks to this so that a nonlinear controller design problem could be transformed to a convex optimization problem.

The following definitions and lemmas are of utmost importance to provide the main contribution of the paper: The convex hull of points of such $c^1, c^2, \dots, c^r$, can be defined as, $\mathrm{co}\{c^i \,:\, i \in [1,r]\} := \left\{ \sum_{i=1}^{r} \alpha_i c^i \,:\, \sum_{i=1}^{r} \alpha_i = 1\,,\; \alpha_i \ge 0 \right\}$.

Lemma 1 *[8]. Let us assume that $T_i \in \mathcal{T}$, is $m\times m$ dimensional, and diagonal matrix whose diagonal entries are either 1 or 0. Besides,we define $T_i^- := I - T_i$ where I stands for an identity matrix in appropriate dimension. Evidently, the set $\mathcal{T}$ has 2^m different elements and it is obvious that if $T_i \in \mathcal{T}$ then $T_i^- \in \mathcal{T}$.*

Lemma 2 *[6] Let $u, v \in \mathbb{R}^m$ are given. Suppose that $|v_i| \le 1 \; \forall i \in [1,m]$, then*

$$\mathrm{sat}(u) \in \mathrm{co}\{T_i u + T_i^- v \;:\; i \in [1, 2^m]\}. \tag{3}$$

Corollary 3 *Let us assume that $F, D \in \mathbb{R}^{m\times n}$ are feedback gain matrices and, $\|Dx\|_\infty \le 1$. Then, $\mathrm{sat}(Fx) \in \mathrm{co}\left\{T_i Fx + T_i^- Dx \;:\; i \in [1, 2^m]\right\}$.*

Notation In this study, standard notation is used, $P = P^{\mathrm{T}} \succ 0$ ($P = P^{\mathrm{T}} \succeq 0$) represents strictly positive Hermitian matrices. The rows of $D \in \mathbb{R}^{m\times n}$ are shown as D_i, $i = 1,\dots,m$. Symmetric blocks induced by off-diagonal elements of a symmetric matrix are labeled as $*$. Besides, $*$ also stands for the transposed version of an unsymmetrical matrix. For example $X + X^{\mathrm{T}} = X + *$. If $\mathbb{R}$ denotes the set of real numbers, $p\times n$ dimensional real matrix is symbolized with $\mathbb{R}^{p\times n}$. Finally, while $I_{r\times r}$, represents $r\times r$ dimensional identity matrix, 0 represents appropriate dimensional null matrix.

4. Main results

Theorem 4 *For a given control law $u(t) = Fx(t)$, consider the system given in (1). Also, assume that disturbance, w, penetrates from the set of $\mathcal{W}$ which is bounded by $1/\delta$. For given gain, γ and constant $\eta > 0$, if feasible $P = P^{\mathrm{T}} \succeq 0$ and $D \in \mathbb{R}^{m\times n}$ matrices can be found satisfying*

$$PA_i + A_i^{\mathrm{T}}P + \frac{1}{\eta}PB_2B_2^{\mathrm{T}}P + \frac{1}{\gamma^2}C^{\mathrm{T}}C \preceq 0 \quad \forall i \in [1, 2^m] \tag{4}$$

and

$$\mathcal{E}(P, \frac{\eta}{\delta} + 1) \subset \mathcal{L}(D) \quad \forall t \ge 0 \qquad \mathcal{E}(S, \varrho) \subset \mathcal{E}(P, 1) \tag{5}$$

where $A_i := (A + B_1(T_i F + T_i^- D))$*, then the closed-loop system will be bounded gain* $\mathcal{L}_2$ *stable from input* w *to output* z *for every initial conditions,* $x(0) \in \mathcal{E}(P,1)$*. Moreover,* $\|z\|^2 \leq \gamma^2\left(\frac{\eta}{\delta}+1\right) \quad \forall t \geq 0$ *and state trajectories that start from inside the ellipsoid* $\mathcal{E}(P,1)$ *will always remain inside* $\mathcal{E}(P,\frac{\eta}{\delta}+1)$ *ellipsoid for every* $w \in \mathcal{W}$*.*

Proof Let us choose a Lyapunov function candidate as $V = x^{\mathrm{T}}Px$. Then, in the light of definition of $\mathcal{E}$, $\forall x \in \mathcal{E}(P,\frac{\eta}{\delta}+1))$,

$$\begin{aligned} \dot{V} &= \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = 2x^{\mathrm{T}}P[Ax + B_1\mathrm{sat}(Fx) + B_2 w] \\ &\leq \max_{i\in[1,2^m]} 2x^{\mathrm{T}}P[Ax + B_1(T_iF + T_i^- D)x + B_2 w]. \end{aligned} \tag{6}$$

On the other hand, for any positive scalar η, $x^{\mathrm{T}}PB_2w + w^{\mathrm{T}}B_2^{\mathrm{T}}Px \leq \frac{1}{\eta}x^{\mathrm{T}}PB_2B_2^{\mathrm{T}}Px + \eta w^{\mathrm{T}}w$. Then assuming (4) is satisfied, $\dot{V} \leq -\frac{1}{\gamma^2}x^{\mathrm{T}}C^{\mathrm{T}}Cx + \eta w^{\mathrm{T}}w \quad \forall t \geq 0$. Integrating both sides from 0 to t, one obtains

$$V(x(t)) \leq -\frac{1}{\gamma^2}\int_0^t z(\tau)^{\mathrm{T}}z(\tau)d\tau + \eta\int_0^t w(\tau)^{\mathrm{T}}w(\tau)d\tau + 1 \quad \forall t \geq 0. \tag{7}$$

Here, since disturbance, w satisfies $w \in \mathcal{W}$ condition and (5) is valid then taking the limit $t \to \infty$ let us write $V(x(t)) \leq \frac{\eta}{\delta}+1 \quad \forall t \geq 0$, which is equivalent to $x(t) \in \mathcal{E}(P,\frac{\eta}{\delta}+1)$, $\forall t \geq 0$. Besides, since through (7), $\forall t \geq 0$ $V(x(t)) \geq 0$, one obtains $\|z\|^2 \leq \gamma^2\left(\frac{\eta}{\delta}+1\right) \quad \forall t \geq 0$. which concludes the proof. □

Remark 5 *If the disturbance acting on the system is equivalent to zero* $\forall t \geq 0$ *and if the negative condition in* $\dot{V} \leq -\frac{1}{\gamma^2}x^{\mathrm{T}}C^{\mathrm{T}}Cx + \eta w^{\mathrm{T}}w$ *is replaced with its strict counterpart, then we obtain* $\dot{V}(x(t)) < -\frac{1}{\gamma^2}\|z\|_2^2 \leq 0,\ \forall x(t) \in \mathcal{E}(P,1)$*. This result leads to a conclusion that all trajectories of* $x(t)$ *originating inside ellipsoid* $\mathcal{E}(P,1)$*, will asymptotically converge to* $x = 0$ *for* $t \to \infty$*. Based on this finding, it should be remarked that inside of ellipsoid* $\mathcal{E}(P,1)$ *will be the region of attraction for the origin.*

Theorem 6 *(Optimal controller synthesize of minimizing* $\mathcal{L}_2-$ *gain) Let us assume that* $S = S^{\mathrm{T}} \succ 0 \in \mathbb{R}^{n\times n}$*,* $\eta > 0$ *and* $\varrho > 0$ *are given. Also let all initial conditions are inside* $\mathcal{E}(P,1)$*. Then,*

$$\min_{Q\succ 0, Y, Z} \gamma$$

$$\begin{pmatrix} AQ + B_1T_iZ + B_1T_i^-Y + * & B_2 & QC^{\mathrm{T}} \\ B_2^{\mathrm{T}} & -\eta I & 0 \\ CQ & 0 & -\gamma^2 I \end{pmatrix} \preceq 0 \qquad \forall i \in [1,2^m] \tag{8}$$

$$\begin{pmatrix} \frac{\delta}{\eta+\delta} & Y_i \\ Y_i^{\mathrm{T}} & Q \end{pmatrix} \succeq 0 \quad \forall i \in [1,m] \tag{9}$$

$$\begin{pmatrix} \frac{1}{\varrho}S & I \\ I & Q \end{pmatrix} \succeq 0. \tag{10}$$

If there exist feasible solutions $Q = Q^{\mathrm{T}} \succ 0$*,* Y*,* Z *to the optimization problem, then* $\mathcal{L}_2$ *gain of the closed-loop system which is obtained by the control law,* $u = ZQ^{-1}x$*, will be always less than* $\gamma\sqrt{\eta+\delta}$*.*

Proof Assume (4) holds. Then, pre- and post multiplying (4) by $Q \triangleq P^{-1}$ yields,

$$Q(A+B_1T_iF)^{\mathrm{T}}+(A+B_1T_iF)Q+B_1T_i^-DQ+(B_1T_i^-DQ)^{\mathrm{T}}+\frac{1}{\eta}B_2B_2^{\mathrm{T}}+\frac{1}{\gamma^2}QC^{\mathrm{T}}CQ\preceq 0$$
$$\forall i\in[1,2^m] \tag{11}$$

By the use of definitions $Z\triangleq FQ$ and $Y\triangleq DQ$, one can rewrite (11) as

$$AQ+B_1T_iZ+QA^{\mathrm{T}}+Z^{\mathrm{T}}T_iB_1^{\mathrm{T}}+B_1T_i^-Y+YT_i^-B_1^{\mathrm{T}}+\frac{1}{\eta}B_2B_2^{\mathrm{T}}+\frac{1}{\gamma^2}QC^{\mathrm{T}}CQ\preceq 0$$
$$\forall i\in[1,2^m]. \tag{12}$$

Applying the Schur complement formula [9], immediately leads to the inequality (8). It is obvious that

$$\mathcal{E}(P,\frac{\eta}{\delta}+1)\subset\mathcal{L}(D)\quad\forall t\ge 0\Leftrightarrow D_i^{\mathrm{T}}D_i\preceq P\left(\frac{\delta}{\eta+\delta}\right)\quad\forall t\ge 0\quad\forall i\in[1,m]$$
$$\Leftrightarrow\left(\frac{\eta+\delta}{\delta}\right)D_i^{\mathrm{T}}D_i\preceq P\quad\forall t\ge 0\quad\forall i\in[1,m]\Leftrightarrow\begin{pmatrix}P & D_i^{\mathrm{T}}\\ D_i & \frac{\delta}{\eta+\delta}I\end{pmatrix}\succeq 0\quad\forall t\ge 0,\quad\forall i\in[1,m]$$
$$\Leftrightarrow\frac{\delta}{\eta+\delta}I\succeq D_iP^{-1}D_i^{\mathrm{T}}\quad\forall t\ge 0,\quad\forall i\in[1,m]$$
$$\Leftrightarrow\frac{\delta}{\eta+\delta}I-D_iQQ^{-1}QD_i^{\mathrm{T}}\succeq 0\;\forall t\ge 0,\forall i\in[1,m]$$
$$\Leftrightarrow\begin{pmatrix}\frac{\delta}{\eta+\delta} & Y_i\\ Y_i^{\mathrm{T}} & Q\end{pmatrix}\succeq 0\quad\forall t\ge 0,\quad\forall i\in[1,m]. \tag{13}$$

Moreover, if we let $\mathcal{E}(S,\varrho)\subset\mathcal{E}(P,1)$ so that the ellipsoid $\mathcal{E}(S,1)$ includes all initial conditions, then it is clear that;

$$\mathcal{E}(S,\varrho)\subset\mathcal{E}(P,1)\Leftrightarrow\frac{1}{\beta}S\succeq P. \tag{14}$$

Applying Schur complement formulation yields;

$$\mathcal{E}(S,\varrho)\subset\mathcal{E}(P,1)\Leftrightarrow\begin{pmatrix}S\frac{1}{\beta} & I\\ I & Q\end{pmatrix}\succeq 0. \tag{15}$$

It is concrete that in order to enlarge set of initial conditions, set of $\mathcal{E}(P,1)$ has to be enlarged first. But, it is only possible by enlarging set of $\mathcal{E}(S,\varrho)$. Meaning, the set of $\mathcal{E}(P,1)$ will be enlarged by the result of enlarging the set of $\mathcal{E}(S,\varrho)$.

Among the other parameters acting on $\mathcal{L}_2$ gain of the system, η not only can be simply chosen as very small positive constant but also chosen as optimization parameter. In this case it is obvious that the degree of freedom of the optimization problem will be increased.

One approach to enlarge the set of $\mathcal{E}(S,\varrho)$ which includes initial conditions, is to increase directly constant β without enlarging the volume of the ellipsoid. To do so, assuming $S=S^{\mathrm{T}}$ is large enough and known positive definite matrix, and constant $\eta>0$ then;

$$\min_{c>0,Q\succ 0,Y,Z}\;(\sigma_1 r+\sigma_2 c)$$

$$\begin{pmatrix} AQ + B_1T_iZ + B_1T_i^{-}Y + * & B_2 & QC^{\mathrm{T}} \\ B_2^{\mathrm{T}} & -\eta I & 0 \\ CQ & 0 & -rI \end{pmatrix} \preceq 0$$
$$\forall i \in [1, 2^m] \tag{16}$$

$$\begin{pmatrix} \frac{\delta}{\eta+\delta} & Y_i \\ Y_i^{\mathrm{T}} & Q \end{pmatrix} \succeq 0 \quad \forall i \in [1, m] \tag{17}$$

$$\begin{pmatrix} cS & I \\ I & Q \end{pmatrix} \succeq 0 \tag{18}$$

optimization problem should be solved along with it.

Here, it is defined that $c := 1/\varrho$. For the sake of simplicity, S can be chosen as unity sphere ($x = 0$ centered circle in two dimension). On the other hand, σ_1, σ_2 are weighting of optimization problem which are nonnegative constants and can be tuned by controller designer accordingly.

5. $\mathcal{L}_2$ control of LPV systems with saturating actuators

Now, consider the following LPV system with a saturated control input,

$$\begin{aligned} \dot{x} &= A(\rho(t))x + B_1(\rho(t))\mathrm{sat}(u) + B_2(\rho(t))w \\ z &= C(\rho(t))x \qquad x(0) = x_0 \end{aligned} \tag{19}$$

where online measurable or computable parameter vector $\rho(t) = \begin{bmatrix} \rho_1(t) & \cdots & \rho_q \end{bmatrix}^{\mathrm{T}}$ and $\dot{\rho}(t) = \begin{bmatrix} \dot{\rho}_1(t) & \cdots & \dot{\rho}_q \end{bmatrix}^{\mathrm{T}}$ vary in compact parameter sets $\mathcal{R}$ and $\mathcal{D}$, respectively where these sets are covered by hyper-rectangles formed by the vertices of parameter bounds as $\mathcal{R}_{vex} = \left\{ \rho_i^v \,:\, \rho_i^v = \bar{\rho}_i \text{ or } \rho_i^v = \underline{\rho_i}^v,\ \forall i = 1, \ldots, q \right\}$ and $\mathcal{D}_{vex} = \left\{ \dot{\rho}_i^v \,:\, \dot{\rho}_i^v = \bar{\dot{\rho}}_i \text{ or } \dot{\rho}_i^v = \underline{\dot{\rho}_i}^v,\ \forall i = 1, \ldots, q \right\}$. Also assume that the system we consider has state-space matrices as affine parameter functions, *i.e.*, $A(\rho) = A_0 + \rho_1 A_1 + \cdots + \rho_q A_q$, $B_1(\rho) = B_{10} + \rho_1 B_{11} + \cdots + \rho_q B_{1q}$, $B_2(\rho) = B_{20} + \rho_1 B_{21} + \cdots + \rho_q B_{2q}$ and $C(\rho) = C_0 + \rho_1 C_1 + \cdots + \rho_q C_q$. Then our goal is to find a suitable state-feedback control law in the form of $u(\rho(t), x(t)) = F(\rho(t))x(t)$ so that the closed-loop system exhibits the control performance criteria listed in Section 1 for all $(\rho(t), \dot{\rho}(t)) \in (\mathcal{R}_{vex} \times \mathcal{D}_{vex})$. The following theorem is the direct extension of Theorem 4 to the LPV systems in the form of (19).

Theorem 7 *Consider the system given in (19). For given positive scalars γ, η and control law $u(\rho(t), x(t)) = F(\rho(t))x(t)$, the closed-loop system is finite-gain $\mathcal{L}_2$ stable for any initial condition $x(0) \in \mathcal{E}(P(\rho), 1)$ and any disturbance $w \in \mathcal{W}$ if there exist $P(\rho) = P^T(\rho) \succ 0$ in the form of $P(\rho) = P_0 + \sum_{i=1}^{q} \rho_i P_i$ and matrix $D(\rho)$ in the form of $D(\rho) = D_0 + \sum_{i=1}^{q} \rho_i D_i$ such that*

$$\begin{aligned} &P(\rho)(A(\rho) + B_1(\rho)(T_iF(\rho) + T_i^{-}D(\rho))) + (A(\rho) + B_1(\rho)(T_iF(\rho) + T_i^{-}D(\rho)))^{\mathrm{T}}P(\rho) \\ &+ \frac{1}{\eta}P(\rho)B_2(\rho)B_2^{\mathrm{T}}(\rho)P(\rho) + \dot{P}(\dot{\rho}) + \frac{1}{\gamma^2}C^{\mathrm{T}}(\rho)C(\rho) \prec 0 \ \forall i \in [1, 2^m]\, \forall \rho \in \mathcal{R}, \forall \dot{\rho} \in \mathcal{D} \end{aligned} \tag{20}$$

and $\mathcal{E}(P(\rho), \frac{\eta}{\delta}+1) \subset \mathcal{L}(D(\rho)) \quad \forall t \geq 0, \forall \rho \in \mathcal{R}$. *If such a feasible solution set is obtained, then*

$$\|z\|^2 \leq \gamma^2 \left(\frac{\eta}{\delta}+1\right) \quad \forall t \geq 0 \tag{21}$$

and any state-trajectory starting inside $\mathcal{E}(P(\rho), 1)$ *always stays inside the ellipsoid* $\mathcal{E}(P(\rho), \frac{\eta}{\delta}+1)$

Proof Similar to the proof of Theorem 4, and thus it is omitted. □

Note that the matrix inequality conditions given in the previous theorem are not in the form of LMIs. Besides, the problem requires infinite-dimensional search with respect to its variables. Hence, the conditions in the previous theorem need to be reduced to finite-dimensional LMI conditions. Next theorem allows us to obtain a sub-optimal $\mathcal{L}_2$ controller by a search in infinite-dimensional parameter space.

Theorem 8 *Given* $S = S^{\mathrm{T}} \succ 0 \in \mathbb{R}^{n\times n}$, $\eta > 0$ *and* $\varrho > 0$, *For all initial conditions which reside in* $\mathcal{E}(S,1)$, *if there exists a feasible solution set* $\{Q(\rho) = Q^{\mathrm{T}}(\rho) \succ 0,\ Y(\rho),\ Z(\rho)\}$, *each in the form of* $Q(\rho) = Q_0 + \sum_{i=1}^{q} \rho_i Q_i$, $Y(\rho) = Y_0 + \sum_{i=1}^{q} \rho_i Y_i$, $Z(\rho) = Z_0 + \sum_{i=1}^{q} \rho_i Z_i$, *satisfying the following optimization problem*

$$\min_{Q(\rho), Y(\rho), Z(\rho)} \gamma^2$$

$$\begin{pmatrix} A(\rho)Q(\rho) + B_1(\rho)T_i Z(\rho) + B_1(\rho)T_i^{-} Y(\rho) - \dot{Q}(\dot{\rho}) + * & B_2(\rho) & Q(\rho)C^{\mathrm{T}}(\rho) \\ B_2^{\mathrm{T}}(\rho) & -\eta I & 0 \\ * & 0 & -\gamma^2 I \end{pmatrix} \preceq 0$$

$$\forall i \in [1, 2^m]\, \forall(\rho, \dot{\rho}) \in \mathcal{R} \times \mathcal{D} \tag{22}$$

$$\begin{pmatrix} \frac{\delta}{\eta+\delta} & i^{th} \text{row of } Y(\rho) \\ * & Q(\rho) \end{pmatrix} \succeq 0 \quad \forall i \in [1, m]\, , \rho \in \mathcal{R} \tag{23}$$

$$\begin{pmatrix} \frac{1}{\varrho} S & I \\ I & Q(\rho) \end{pmatrix} \succeq 0\ \forall \rho \in \mathcal{R} \tag{24}$$

then the closed-loop system formed by the LPV control law $u(t) = Z(\rho)Q^{-1}(\rho)x(t)$, *always exhibits an closed-loop* $\mathcal{L}_2$ *gain less than* $\gamma\sqrt{\eta+\delta}$.

Proof Since the proof is very similar to the proof Theorem 6, it is omitted. □

Utilizing the multi-convexity argument proposed in [10], Theorem 8 can be treated as a finite-dimensional LMI optimization problem as follows:

$$\min_{c>0,\, Q(\rho),\, Y(\rho),\, Z(\rho)} (\sigma_1 r + \sigma_2 c)$$

$$\begin{pmatrix} A(\rho)Q(\rho) + B_1(\rho)T_i Z(\rho) + B_1(\rho)T_i^{-} Y(\rho) - \dot{Q}(\dot{\rho}) + * & B_2(\rho) & Q(\rho)C^{\mathrm{T}}(\rho) \\ B_2^{\mathrm{T}}(\rho) & -\eta I & 0 \\ * & 0 & -rI \end{pmatrix} \preceq 0$$

$$\forall i \in [1, 2^m]\, \forall(\rho, \dot{\rho}) \in \mathcal{R}_{vex} \times \mathcal{D}_{vex} \tag{25}$$

$$\begin{pmatrix} \frac{\delta}{\eta+\delta} & i^{th}\text{row of } Y(\rho) \\ * & Q(\rho) \end{pmatrix} \succeq 0 \quad \forall i \in [1, m], \, \rho \in \mathcal{R}_{vex} \tag{26}$$

$$\begin{pmatrix} cS & I \\ I & Q(\rho) \end{pmatrix} \succeq 0 \; \forall \rho \in \mathcal{R}_{vex} \tag{27}$$

$$\begin{pmatrix} A_j Q_j + B_{1_j} T_i Z_j + B_{1_j} T_i^- Y_j + * & Q_j C_j^{\mathrm{T}} \\ * & 0 \end{pmatrix} \succeq 0$$
$$\forall i \in [1, 2^m], \, \forall j = [1, \ldots, q] \tag{28}$$

where $c := 1/\varrho$.

Note that (28) comes from the multi-convexity argument.

6. Simulation studies

To demonstrate effectiveness of the proposed theory, single track vehicle dynamics is modeled as an application, and it has been focused on minimizing problem of $\mathcal{L}_2$ gain from disturbances to performance outputs, z [11], [12].

Assuming $\dot{v} = 0$, $x \cos \delta = 0$, $x \sin \delta = 0$ the single track vehicle model [13], [14] is obtained as

$$\begin{aligned} Mv(\dot{\beta} + \dot{\psi}) &= c_F \Pi_1 + c_R \Pi_2 \\ J_Z \ddot{\psi} &= c_F L_F \Pi_1 - c_R L_R \Pi_2. \end{aligned} \tag{29}$$

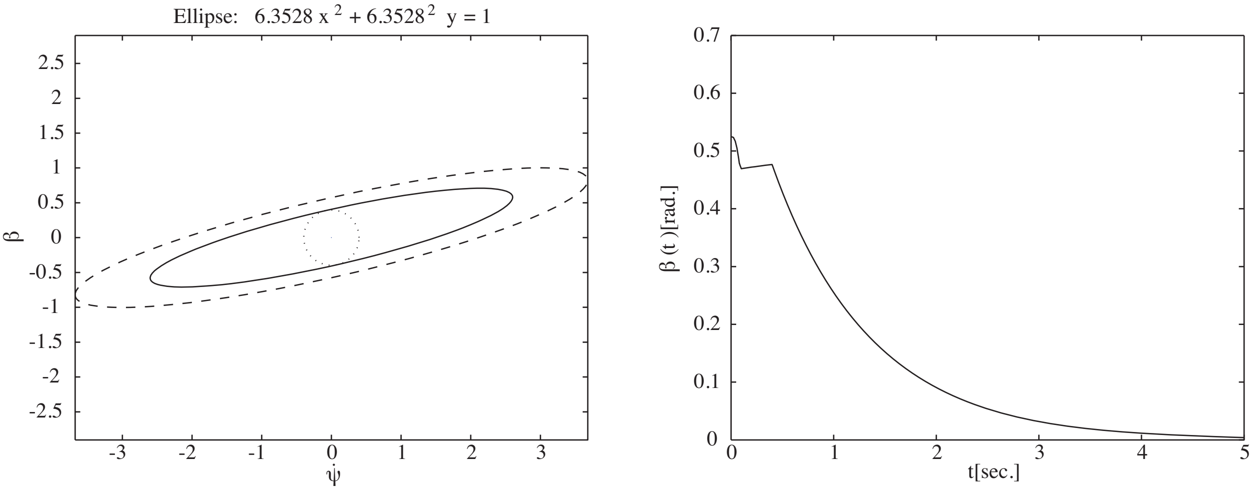
where $\Pi_1 = \left(\delta_W - \beta - \frac{L_F \dot{\psi}}{v}\right)$ and $\Pi_2 = \left(-\beta + \frac{L_R \dot{\psi}}{v}\right)$. Here, δ_W represents wheel turning-angle in radians. The other terms are; $c_F = c_R = 34377\mathrm{N/rad}$, $L_F = 1.4\mathrm{m}$, $L_R = 1.7\mathrm{m}$, $M = 1500\mathrm{kg}$ and $k_G = 1.3\mathrm{m}$ is turning radius. Accordingly, the inertia in the direction of z axis is defined as $J_Z = Mk_G^2$. During the simulation studies for the non-LPV model, reverse driving is considered and vehicle speed is assumed to be constant at $v = -30\mathrm{km/hr}$. While positive speed is considered as vehicle in forward direction and stable, negative speed is considered as vehicle in backward direction and always unstable. In this study, disturbances acting on system are assumed to be sensor noises added on $\dot{\beta}$. Controlled outputs are β and $\dot{\psi}$. Wheel angle is assumed to be as control input, δ_W, and it is assumed to be bounded as $-\pi/4 \leq \delta_W \leq \pi/4$.

During computer simulations which are carried out under MATLAB, it is assumed that $\eta = 1$ and disturbance signal $w = \sqrt{2.5}(s(t) - s(t-0.4))$ is used. Here Here $s(t)$ is the well-known unit step function. Choosing state variables as, $x = \begin{pmatrix} \dot{\psi} & \beta \end{pmatrix}^{\mathrm{T}}$ matrices for generalized system are;

$$A = \begin{pmatrix} 2.192 & 4.068 \\ -0.992 & 1.528 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 18.985 \\ -0.764 \end{pmatrix} \times \frac{\pi}{4}, \; B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C = I_{2\times 2}. \tag{30}$$

The controller design problem for the system is coded through YALMIP interface under MATLAB and solved with SEDUMI solver. The values of the variables at the end of design are obtained as: $\gamma = 3.31$, $\varrho = 0.1574$

Under these circumstances, state feedback controller gain vector is obtained to be as $F = \begin{pmatrix} -1835 & 5832.4 \end{pmatrix}$. Accordingly, the ellipsoids are as in Figure 1 where the dashed line represents ellipsoid $\mathcal{E}(S, \varrho)$ which is used for enlarging initial conditions space, $\mathcal{E}(P, 1)$.



(a) EllipsoidalSurfaces.

(b) Trajectoryofclosedloop $\beta(t)$.

Figure 1. Ellipsoidal Surfaces and Trajectory of closed loop $\beta(t)$

Notably, all states originating from set of $\mathcal{E}(P,1)$, solid line, will result in the larger outer ellipsoid $\mathcal{E}(P,\frac{\eta}{\delta}+1)$, dashed line, under unity energy level disturbances. It can be claimed that simulation results are in favor of analytical approach.

Figure 1 demonstrates state trajectory $\beta(t)$. Here, it should be noted that the initial conditions are chosen as $\beta(0)=\pi/6$. Also, the other optimization parameters σ_1 andσ_2 are chosen as $\sigma_1=\sigma_2=1$, $S=I$ and $\eta=1$. If the optimization problem is only focused on enlarging ellipsoid of initial conditions, $\mathcal{E}(P,1)$, then the ellipsoid $\mathcal{E}(P,1)$ is obtained as a larger ellipsoid where $\varrho=0.4665$. But in such case, $\mathcal{L}_2$ gain of the system is get worse ($\gamma=13844$). In short, there is a trade off between enlarging ellipsoid of initial conditions, $\mathcal{E}(P,1)$, and $\mathcal{L}_2$ gain of the system and the range of the trade off is defined by σ_1 and σ_2.

In order to demonstrate the application of LPV counterpart of the proposed theory, we consider the system in (29) where the parameter v is time-varying. In this sense, considering the scheduling parameter vector as $\rho=\begin{bmatrix}1/v & 1/v^2\end{bmatrix}^T$, one obtains the LPV system in state-space form as

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} \frac{-C_F L_F^2 - C_R L_R^2}{J_Z}\rho_1 & \frac{-C_F L_F + C_R L_R}{J_Z} \\ \frac{-L_F C_F + C_R L_R}{M}\rho_2 - 1 & \frac{-C_F - C_R}{M}\rho_1 \end{pmatrix} x + \begin{pmatrix} \frac{C_F L_F}{J_Z} \\ \frac{C_F}{M}\rho_1 \end{pmatrix} \times (\frac{\pi}{4})\mathrm{sat}(u) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \\
z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x
\end{aligned}$$

It is assumed that the vehicle speed varies in the range of $v \in [-10,\,-50][km/h]$. However, we assume that we do not have any information for the bound of dv/dt. Here, we focus only on the minimization of $\mathcal{L}_2$ gain, γ. Applying the multi-objective controller design procedure presented in the previous section by selecting the constants as $\eta=1$, $\delta=1$, $\sigma_1=\sigma_2=1$, the worst case $\mathcal{L}_2$ gain from w to z is obtained as $\gamma=1.55$ under the affect of the same disturbance presented in the previous nominal control example. On the other hand, ϱ is obtained as 0.26. The other semi-definite programming variables are found as

$$Q_0 = \begin{pmatrix} 6.1644 & 0.8188 \\ 0.8188 & 0.4315 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} -0.0337 & 3.2349 \\ 3.2349 & 2.2584 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0.0000 & 0.0001 \\ 0.0001 & -3.5614 \end{pmatrix},$$

$$Z_0 = \begin{pmatrix} -1.0087 & 0.1399 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 20.0581 & 5.8738 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} -85.4383 & 1.6309 \end{pmatrix}.$$

Figure 2 demonstrates the closed-loop state trajectory $\beta(t)$ when the LPV controller

$$u(\rho(t), x(t)) = \big(Z_0 + \sum_{i=1}^{2} \rho_i(t) Z_i\big)\big(Q_0 + \sum_{i=1}^{2} \rho_i(t) Q_i\big)^{-1} x(t)$$

is applied to the system. Again the initial conditions for the states are chosen as $\beta(0) = \pi/6[rad]$ and $\dot{\psi}(0) = 0$. Note that better disturbance attenuation rates and larger initial condition sets are obtained for the LPV controller when compared with the non-LPV counterpart.

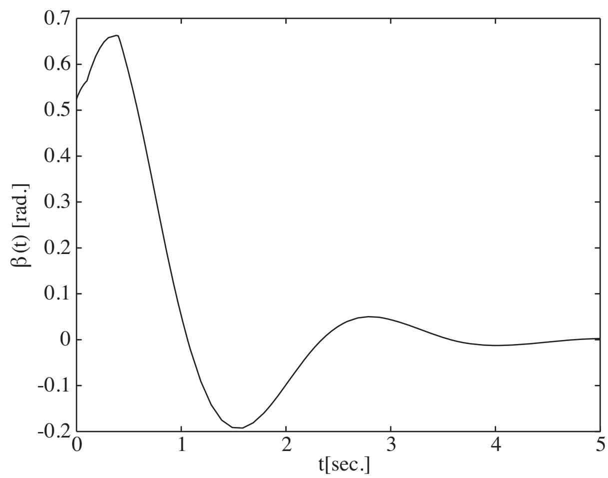


Figure 2. Trajectory of closed loop $\beta(t)$ with multi-objective optimization methodology and LPV control.

7. Conclusions

In this study, an approach to the optimal state-feedback controller design which minimizes $\mathcal{L}_2$ gain from disturbances to performance output for both Linear Time Invariant (LTI) systems and LPV systems having saturating actuators is presented. In addition, the controller design procedure includes enlarging the set of initial conditions. It should be underlined that nonlinear actuator model in the problem is represented with LMIs through convex hull representation of some linear feedback. After presenting theoretical background of the approach, in order to demonstrate the success of optimizing method, simulation studies are given for a single track vehicle model. It has been successfully shown that the proposed method can be used for disturbance rejection problems in such LTI and LPV systems with saturating actuators. For future studies one can try to enlarge the set of disturbances affecting system, $\mathcal{W}$. But, in this case, the optimization problem will be bilinear with respect to its variables. Note that to be able to solve such a problem one may use cone-complementary methods and BMI kind of solvers.

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