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Delay-dependent stability criteria for interval time-varying delay systems with nonuniform delay partitioning approach

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Abstract

This paper investigates the conservatism reduction of Lyapunov-Krasovskii based conditions for the stability of a class of interval time-varying delay systems. The main idea is based on the nonuniform decomposition of the integral terms of the Lyapunov-Krasovskii functional. The delay interval is decomposed into a finite number of nonuniform segments with some scaling parameters. Both differentiable delay case and nondifferentiable delay case and unknown delay derivative bound case are taken into consideration. Sufficient delay-dependent stability criteria are derived in terms of matrix inequalities. Two suboptimal delay fractionation schemes, namely, linearization with cone complementary technique and linearization under additional constraints are introduced in order to find a feasible solution set using LMI solvers with a convex optimization algorithm so that a suboptimal maximum allowable delay upper bound is achieved. It is theoretically demonstrated that the proposed technique has reduced complexity in comparison to some existing delay fractionation methods from the literature. A numerical example with case studies is given to demonstrate the effectiveness of the proposed method with respect to some existing ones from the literature.

Key Words: *Time delay systems, interval time-varying delay, delay partitioning, cone complementary method, linear matrix inequality.*

1. Introduction

Many physical and dynamical systems are inevitably subject to time-delay which usually results from long transmission lines, finite speed of information processing rate and causal nature of systems such that physical systems can not respond abruptly. The existence of time-delay may lead to a degredation in performance and sometimes it may yield an unstable behavior. Time-delay systems are investigated as delay differential systems [1]. The time-delay can be constant or time-varying with differentiable nature where delay derivative bound may be known or unknown or the delay can be with nondifferentiable characteristics. The stability analysis of such systems are conducted in the form of either delay-independent or delay-dependent [2]–[5].

In general, time-delay is assumed to vary in an interval whose lower bound is usually considered to be zero. However, in recent years investigation of time-delay systems in which the time-delay possesses a nonzero lower bound is taken into consideration [6]–[16]. Introducing artificially fractions of the time-delay, delay-dependent stability results are obtained for retarded systems which lead to a sequence of LMI conditions of growing dimensions that result in decreasing conservatism [17]. A partitioning scheme for the time-varying delay is introduced in [18] for constructing Lyapunov-Krasovskii functional which takes into account the whole state of the time-delay system. Finally, another delay partitioning projection approach is presented in [19] for stability analysis of neutral systems. Unfortunately, the authors who consider delay fractionation approach do not introduce any technique that describes how the delay interval is partitioned. They simply adopt to use a uniform delay partitioning scheme. Inspired by the aforementioned work, we argue if a generalized and nonuniform delay-partitioning method can be used for the stability problem of interval time-varying delay systems which results to be the motivation for the present work

In this paper, we study the stability problem of a class of time-delay systems with interval time-varying delay. We propose a computational way of improving the design of Lyapunov-Krasovskii functionals. The idea is to split the Lyapunov-Krasovskii functional integrals over several time intervals (subintervals of the delay interval). The same idea is already present in [20] on Lyapunov-Krasovskii functional discretization, but, here the sizes of splitting intervals are taken as parameters to be tuned. Some improved stability criteria are formulated in terms of matrix inequalities to handle differentiable and nondifferentiable time-varying delays. The scalar parameters used for delay partitioning are considered to be decision variables. Two suboptimal delay fractionation schemes, namely, linearization with cone complementary technique and linearization under additional constraints are introduced so that a suboptimal maximum allowable delay upper bound is achieved. In particular, first a cone complementary problem is presented so that a nonlinear minimization problem with LMI conditions replaces the derived nonconvex feasibility problem [21]. Employing the linearization method allows to find a suboptimal maximal delay along with the scalar delay partitioning parameters. Second, some additional bounding constraints are imposed on the delay partitioning parameters such that the original matrix inequality is converted into an LMI. Moreover, it is theoretically demonstrated that the proposed technique has reduced complexity in comparison to some existing delay fractionation methods from the literature. A numerical example is introduced with case studies to illustrate the application of the proposed method. An earlier preliminary form of the developed results can be found in [22].

2. Problem statement

Let us consider a class of interval time-varying delay systems given by

$$\dot{x}(t) = Ax(t) + A_h x(t - h(t)) \tag{1}$$

$$x(t) = \Phi(t), \quad \forall t \in [-h_2, 0] \tag{2}$$

where $x(t) \in \Re^n$ is the state vector of the system, $A \in \Re^{n\times n}$, $A_h \in \Re^{n\times n}$ are constant system matrices, $h(t)$ is the time-varying delay satisfying

$$h_1 \leq h(t) \leq h_2 \tag{3}$$

where h_1, h_2 are positive scalar constants. If the time-varying delay is of differentiable nature, then we have

$$|\dot{h}(t)| \leq \mu \tag{4}$$

where μ is a positive scalar constant. Moreover, $\Phi(t)$ is the initial condition function which is a continuous vector-valued function of $t \in [-h_2, 0]$. Adopting a nonuniform delay partitioning approach, we interpret the delay interval $[-h_1, 0]$ as the union of subintervals

$$[-h_1, 0] = [-h_1, -\alpha_N h_1] \bigcup [-\alpha_N h_1, -\alpha_{N-1} h_1] \bigcup \ldots \bigcup [-\alpha_2 h_1, -\alpha_1 h_1] \bigcup [-\alpha_1 h_1, 0] \tag{5}$$

where the delay partitioning parameters α_i, $i = 1, \ldots, N$ are some scalar constants to be selected in accordance with

$$0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{N-1} \leq \alpha_N \leq \tag{6}$$

The primary objective of the present work is to develop some sufficient stability criteria based on a Lyapunov-Krasovskii functional which will be chosen with respect to the subinterval pattern given in Eqn. (5). The secondary goal is to interpret the delay partitioning parameters α_i, $i = 1, \ldots, N$ as decision variables so that they can be optimized in order to achieve a maximum allowable delay upper bound.

3. Main results

The following theorem presents a sufficient stability criterion for interval time-varying delay system Eqn. (1), Eqn. (2).

Theorem 1 (21) *Given scalar constants h_1, h_2 satisfying Eqn. (3) and a positive scalar μ, the interval time-varying delay system Eqn. (1), Eqn. (2) is guaranteed to be globally asymptotically stable if there exist symmetric matrices $P^T = P > 0$, $Q_i^T = Q_i \geq 0$, $R^T = R \geq 0$, $S_i^T = S_i > 0$, $T^T = T > 0$, $U^T = U > 0$, and scalar parameters α_i, $i = 1, \ldots, N$ satisfying Eqn. (6) and*

$$\Omega = \begin{bmatrix} \Omega_{11} & Q_1 & \cdots & 0 & 0 & 0 & \Omega_{1(N+4)} \\ * & \Omega_{22} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \Omega_{(N+1)(N+1)} & Q_{N+1} & 0 & 0 \\ * & * & \cdots & * & \Omega_{(N+2)N+2)} & 0 & R \\ * & * & \cdots & * & * & \Omega_{(N+3)(N+3)} & R \\ * & * & \cdots & * & * & * & \Omega_{(N+4)(N+4)} \end{bmatrix} < 0 \tag{7}$$

where $\Omega_{11} = A^T P + PA - Q_1 + S_1 + A^T Q A$, $\Omega_{1(N+4)} = PA_h + A^T Q A_h$, $\Omega_{22} = -Q_1 - Q_2 - S_1 + S_2$, $\Omega_{(N+1)(N+1)} = -Q_N - Q_{N+1} - S_N + S_{N+1}$, $\Omega_{(N+2)(N+2)} = -Q_{N+1} - R - S_{N+1} + T + U$, $\Omega_{(N+3)(N+3)} = -T - R$, $\Omega_{(N+4)(N+4)} = -2R - (1-\mu)U + A_h^T Q A_h$ with $Q = h_1^2[\alpha_1^2 Q_1 + \sum_{i=2}^{N} (\alpha_i - \alpha_{i-1})^2 Q_i + (1-\alpha_N)^2 Q_{N+1}] + (h_2 - h_1)^2 R$ and (*) represents the symmetric terms.

Proof Let us choose a candidate Lyapunov-Krasovskii functional as follows

$$V(x(t), t) = \sum_{i=1}^{3} V_i \tag{8}$$

where $V_1 = x^T(t)Px(t)$, $V_2 = \alpha_1 h_1 \int_{-\alpha_1 h_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Q_1\dot{x}(s)dsd\theta$
$+\sum_{i=2}^{N}(\alpha_i-\alpha_{i-1})h_1 \int_{-\alpha_i h_1}^{-\alpha_{i-1}h_1} \int_{t+\theta}^{t} \dot{x}^T(s)Q_i\dot{x}(s)dsd\theta + (1-\alpha_N)h_1 \int_{-h_1}^{-\alpha_N h_1} \int_{t+\theta}^{t} \dot{x}^T(s)Q_{N+1}\dot{x}(s)dsd\theta$
$+(h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta, V_3 = \int_{t-\alpha_1 h_1}^{t} x^T(s)S_1x(s)ds + \sum_{i=2}^{N}\int_{t-\alpha_i h_1}^{t-\alpha_{i-1}h_1} x^T(s)S_ix(s)ds$
$+\int_{t-h_1}^{t-\alpha_N h_1} x^T(s)S_{N+1}x(s)ds + \int_{t-h_2}^{t-h_1} x^T(s)Tx(s)ds] + \int_{t-h(t)}^{t-h_1} x^T(s)Ux(s)ds$. Taking the time-derivative of $V(x(t),t)$ along the state trajectory of system Eqn. (1), Eqn. (2) yields

$$\dot{V}(x(t),t) = \sum_{i=1}^{3}\dot{V}_i \tag{9}$$

We can calculate $\dot{V}_1$ as

$$\dot{V}_1 = 2x^T(t)P\dot{x}(t) = x^T(t)(A^TP+PA)x(t) + 2x^T(t)PA_hx(t-h(t)) \tag{10}$$

In a similar manner we can compute $\dot{V}_2$ as follows

$$\begin{aligned}\dot{V}_2 \;=\; & \alpha_1^2h_1^2\dot{x}^T(t)Q_1\dot{x}(t) - \alpha_1h_1\int_{t-\alpha_1h_1}^{t}\dot{x}^T(s)Q_1\dot{x}(s)ds + \sum_{i=2}^{N}[(\alpha_i-\alpha_{i-1})^2h_1^2\dot{x}^T(t)Q_i\dot{x}(t)\\ & -(\alpha_i-\alpha_{i-1})h_1\int_{t-\alpha_ih_1}^{t-\alpha_{i-1}h_1}\dot{x}^T(s)Q_i\dot{x}(s)ds] + (1-\alpha_N)^2h_1^2\dot{x}^T(t)Q_{N+1}\dot{x}(t)\\ & -(1-\alpha_N)h_1\int_{t-h_1}^{t-\alpha_Nh_1}\dot{x}^T(s)Q_{N+1}\dot{x}(s)ds\end{aligned} \tag{11}$$

Note that we can obtain the following inequality

$$\begin{aligned}-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R\dot{x}(s)ds \;\le\; & -[h(t)-h_1]\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)R\dot{x}(s)ds\\ & -[h_2-h(t)]\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)R\dot{x}(s)ds\end{aligned} \tag{12}$$

Substituting Eqn. (12) into Eqn. (11) and applying Jensen integral inequality [11] allows to obtain

$$\begin{aligned}\dot{V}_2 \;\le\; & \dot{x}^T(t)Q\dot{x}(t) - [x(t)-x(t-\alpha_1h_1)]^TQ_1[x(t)-x(t-\alpha_1h_1)]\\ & -\sum_{i=2}^{N}[x(t-\alpha_{i-1}h_1)-x(t-\alpha_ih_1)]^TQ_i[x(t-\alpha_{i-1}h_1)-x(t-\alpha_ih_1)]\\ & -[x(t-h_1)-x(t-h(t))]^TR[x(t-h_1)-x(t-h(t))]\\ & -[x(t-h(t))-x(t-h_2)]^TR[x(t-h(t))-x(t-h_2)\end{aligned} \tag{13}$$

Finally, we can compute $\dot{V}_3$ as follows

$$
\begin{aligned}
\dot{V}_3 \leq\ & x^T(t)S_1x(t) - x^T(t-\alpha_1h_1)S_1x(t-\alpha_1h_1) \\
& + \sum_{i=2}^{N}\left[x^T(t-\alpha_{i-1}h_1)S_ix(t-\alpha_{i-1}h_1) - x^T(t-\alpha_ih_1)S_ix(t-\alpha_ih_1)\right] \\
& + x^T(t-\alpha_Nh_1)S_{N+1}x(t-\alpha_Nh_1) - x^T(t-h_1)S_{N+1}x(t-h_1) \\
& + x^T(t-h_1)Tx(t-h_1) - x^T(t-h_2)Tx(t-h_2) + x^T(t-h_1)Ux(t-h_1) \\
& -(1-\mu)x^T(t-h(t))Ux(t-h(t))
\end{aligned} \tag{14}
$$

Substituting Eqn. (10), Eqn. (13), Eqn. (14) into Eqn. (9) gives

$$
\dot{V}(x(t),t) \leq \chi^T(t)\Omega\chi(t) \tag{15}
$$

where $\chi(t) = \left[\begin{array}{cccccccc} x^T(t) & x^T(t-\alpha_1h_1) & \cdots & x^T(t-\alpha_Nh_1) & x^T(t-h_1) & x^T(t-h_2) & x^T(t-h(t)) \end{array}\right]^T$
Therefore, if the matrix inequality Eqn. (7) is satisfied then we get

$$
\dot{V}(x(t),t) \leq \chi^T(t)\Omega\chi(t) < 0 \tag{16}
$$

implying that system Eqn. (1), Eqn. (2) is guaranteed to be globally asymptotically stable. This completes the proof. □

3.1. A suboptimal delay fractionation: linearization with cone complementary technique

Note that the stability criterion Eqn. (7) is not in the form of convex LMI due to the delay partitioning parameters $\alpha_i,\ i = 1,\ldots,N$ involved in the decision variable Q. However, we can pursue a linearization method for investigating the feasibility problem of Eqn. (7) similar to the approach given in [23]. There exists always a symmetric and positive definite matrix $L^T = L > 0$ such that

$$
\alpha_1^2Q_1 + \sum_{i=2}^{N}(\alpha_i - \alpha_{i-1})^2Q_i + (1-\alpha_N)^2Q_{N+1} \leq L \tag{17}
$$

then by Schur complement [23], Eqn. (17) is equivalent to

$$
\left[\begin{array}{cccccc}
-L & \alpha_1\mathrm{I} & (\alpha_2-\alpha_1)\mathrm{I} & \cdots & (\alpha_N-\alpha_{N-1})\mathrm{I} & (1-\alpha_N)\mathrm{I} \\
* & -Q_1^{-1} & 0 & \cdots & 0 & 0 \\
* & * & -Q_2^{-1} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & \cdots & -Q_N^{-1} & 0 \\
* & * & * & \cdots & * & -Q_{N+1}^{-1}
\end{array}\right] \leq 0 \tag{18}
$$

Introducing new variables $Z_i = Q_i^{-1},\ i = 1,\ldots,N+1$ and utilizing a cone complementary problem leads to propose a nonlinear minimization algorithm with LMI conditions replacing the original nonconvex feasibility problem of Theorem 1, minimize trace $\left(\sum_{i=1}^{N+1} Q_i Z_i\right)$ subject to

$$\begin{bmatrix} -L & \alpha_1 \mathrm{I} & (\alpha_2-\alpha_1)\mathrm{I} & \cdots & (\alpha_N-\alpha_{N-1})\mathrm{I} & (1-\alpha_N)\mathrm{I} \\ * & -Z_1 & 0 & \cdots & 0 & 0 \\ * & * & -Z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & -Z_N & 0 \\ * & * & * & \cdots & * & -Z_{N+1} \end{bmatrix} < 0$$

$$\begin{bmatrix} Q_i & \mathrm{I} \\ * & Z_i \end{bmatrix} \geq 0, \quad i = 1, \ldots, N+1 \tag{19}$$

$$\begin{bmatrix} \bar{\Omega}_{11} & Q_1 & \cdots & 0 & 0 & 0 & \Omega_{1(N+4)} \\ * & \Omega_{22} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \Omega_{(N+1)(N+1)} & Q_{N+1} & 0 & 0 \\ * & * & \cdots & * & \Omega_{(N+2)N+2)} & 0 & R \\ * & * & \cdots & * & * & \Omega_{(N+3)(N+3)} & R \\ * & * & \cdots & * & * & * & \Omega_{(N+4)(N+4)} \end{bmatrix} < 0$$

where $\bar{\Omega}_{11} = A^T P + PA - Q_1 + S_1 + A^T Q_0 A$ with $Q_0 = h_1^2 L + (h_2 - h_1)^2 R$ and the remaining entries are same as described in the statement of Theorem 1.

Remark 1 *The idea of dividing the time-varying delay into a constant part and a varying function is not new and already studied in the literature as mentioned in the section of introduction. Moreover, in [18],[19],[25] and [26], a fractionation scheme for the constant delay has been developed. The present work extends the delay partitioning approach to the stability analysis of interval time-varying delay systems. As a matter of the fact, the original contribution arises from the extension to a nonuniform fractionation of the constant delay. This is achieved by adding appropriate functional terms with respect to the constant delay fractions and by introducing another functional element which copes with the time-varying delay. In addition, a linearization technique is proposed to tune the fractionation scheme automatically, thus leading to obtain suboptimal results.*
When the information of the time-derivative of time-varying delay is unknown or if the time-varying delay is nondifferentiable, we choose $U = 0$ and present the following result derived from Theorem 1.

Corollary 1 *Given scalar constants h_1, h_2 satisfying Eqn. (3), the interval time-varying delay system Eqn. (1), Eqn. (2) is guaranteed to be globally asymptotically stable if there exist symmetric matrices $P^T = P > 0$, $Q_i^T = Q_i \geq 0$, $R^T = R \geq 0$, $S_i^T = S_i > 0$, $T^T = T > 0$, and scalar parameters α_i, $i = 1, \ldots, N$ satisfying Eqn. (6) and*

$$\bar{\Omega} = \begin{bmatrix} \Omega_{11} & Q_1 & \cdots & 0 & 0 & 0 & \Omega_{1(N+4)} \\ * & \Omega_{22} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \Omega_{(N+1)(N+1)} & Q_{N+1} & 0 & 0 \\ * & * & \cdots & * & \bar{\Omega}_{(N+2)N+2)} & 0 & R \\ * & * & \cdots & * & * & \Omega_{(N+3)(N+3)} & R \\ * & * & \cdots & * & * & * & \bar{\Omega}_{(N+4)(N+4)} \end{bmatrix} < 0 \tag{20}$$

where $\bar{\Omega}_{(N+2)(N+2)} = -Q_{N+1} - R - S_{N+1} + T$, $\bar{\Omega}_{(N+4)(N+4)} = -2R + A_h^T Q A_h$ *and the remaining entries are same as described in the statement of Theorem 1.*

3.2. Another suboptimal delay fractionation: introducing additional constraints

We can choose the delay partitioning parameters α_i, $i = 1, \ldots, N$ in such a way that

$$0 \le \alpha_1 \le \beta, 0 \le \alpha_i - \alpha_{i-1} \le \beta, i = 2, \ldots, N, 0 \le 1 - \alpha_N \le \beta \tag{21}$$

where β is a scalar constant satisfying $0 \le \beta \le 1$. Elaborating the constraints imposed in Eqn. (21) yields

$$\begin{array}{cccc} \alpha_1 \le \beta, & \alpha_2 - \alpha_1 \le \beta & \Rightarrow & \alpha_2 \le \beta + \alpha_1 \le \beta + \beta = 2\beta \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{N-1} \le (N-1)\beta, & \alpha_N - \alpha_{N-1} \le \beta & \Rightarrow & \alpha_N \le \beta + \alpha_{N-1} \le \beta + (N-1)\beta = N\beta \\ \alpha_N \le N\beta, & 1 - \alpha_N \le \beta & \Rightarrow & 1 \le \beta + \alpha_N \le \beta + N\beta = (N+1)\beta \end{array} \tag{22}$$

It follows from Eqn. (22) that β needs to be chosen in accordance with

$$\frac{1}{N+1} \le \beta \le 1 \tag{23}$$

Moreover, the inequality Eqn. (21) requires Q satisfy

$$Q \le h_1^2 \left(\beta^2 Q_1 + \sum_{i=2}^{N} \beta^2 Q_i + \beta^2 Q_{N+1} \right) + (h_2 - h_1)^2 R = h_1^2 \beta^2 \sum_{i=2}^{N+1} Q_i + (h_2 - h_1)^2 R \tag{24}$$

Since the stability criterion given in Eqn. (7) has symmetric and strictly positive definite terms resulting from the quadratic element of $h_1^2 \dot{x}^T(t) Q \dot{x}(t)$, the natural selection is to choose a minimized Q satisfying Eqn. (24) which requires to choose β in view of Eqn. (23) as $\beta = \frac{1}{N+1}$. As a result we find that Q will be satisfying

$$Q \le \frac{h_1^2}{(N+1)^2} \sum_{i=2}^{N+1} Q_i + (h_2 - h_1)^2 R \tag{25}$$

Therefore, replacing Q with the right-hand part of Eqn. (25) in Eqn. (7) leads to obtain a linear matrix inequality form of stability condition which is subject to yield only suboptimal results on the maximum allowable delay upper bound.

Remark 2 *It follows from Eqn. (22) that* $0 < \alpha_i \le i\beta$, $i = 1, \ldots, N$ *where* $\beta = 1/N + 1$. *At first glance, this situation might mislead to the case of uniform partitioning which is not true of course. For example, let* $N = 3$, *then* β *should be chosen as 0.25. Therefore,* $0 < \alpha_1 \le 0.25$, $\alpha_1 \le \alpha_2 \le 0.5$, $\alpha_2 \le \alpha_3 \le 0.75$. *Now, choosing* $\alpha_1 = 0.25$, $\alpha_2 = 0.5$, $\alpha_3 = 0.75$ *implies a uniform partitioning scheme which is not the unique one. Another possible fractionation can be given as* $\alpha_1 = 0.15$, $\alpha_2 = 0.20$, $\alpha_3 = 0.35$ *which shows that infinitely many such selection of partitioning parameters can be obtained and uniform pattern is just one among those infinite choices. Hence, the second suboptimal approach is a more general than the idea of uniform delay fractionation paradigm.*

4. Reduced complexity

In order to demonstrate that the proposed stability criteria yield less conservative results as the number of partitions are increased, one can refer to the approach given in [17]. Moreover, we can also show that the proposed technique has reduced complexity in comparison to the existing delay fractionation methods from the literature. It is reported in [17] that the number of decision variables involved in their existing stability criterion is $N_{e1} = \frac{Nn(Nn+1)}{2}n(n+1)$ where N, n denote the number of delay partitions and degree of time-delay system, respectively. In a similar manner, we can calculate the number of decision variables required for our proposed stability criterion and find that $N_p = \frac{(3N+7)n(n+1)}{2} + N$. It is apparently seen that $N_{e1} < N_p$ for $1 \leq N \leq 4$. However, when $N > 4$, one can see that $N_p < N_{e1}$ for $n \geq 4$. This shows that the method in [17] requires less number of decision variables for only a finite number of cases while the proposed method of this note guarantees to utilize less number of decision variables for infinitely many cases. In a similar manner, we can calculate the number of variables involved in the method given in [19]. We find that $N_{e2} = \frac{(2N+3)n(n+1)}{2} + Nn^2$. Now we can compare N_{e2} and N_p in such a way that $N_p - N_{e2} < 0$ implies $N(-n^2+n+2)+4n^2+4n < 0$. Therefore, $N_p - N_{e2} < 0$ can not be guaranteed for only $n = 2$. One can select N in accordance with $N \geq \frac{4n^2+4n}{n^2-n-2}$ for $n > 2$ which covers infinitely many cases that the proposed method requires less number of decision variables thus with reduced complexity when compared to that of [19].

5. Numerical example

In this section, a numerical example is presented with case studies to demonstrate the application of Theorem 1 and Corollary 1.

Example 1 *We consider an interval time-varying delay system example which is also given in [12],and [16] as follows*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h(t)) \tag{26}$$

Table 1 lists the maximum allowable upper bound (MAUB) for h_2 with respect to the different values of h_1 under various delay-derivative bounds along with some existing results from the literature. It can be seen from Table 1 that the results achieved with the proposed stability criterion of Theorem 1 are less conservative when compared with those obtained in [12], and [16]. Moreover, the conservativeness seems to decrease as N increases. In addition, we assume that the upper bound of the time-derivative of the time-varying delay is unknown or the time-varying delay is nondifferentiable. Table 2 presents the numerical results on the maximum allowable upper bound for the delay, h_2 obtained by the application of Corollary 1 in comparison to the existing results from the literature. It follows from Table 2 that Corollary 1 with the linearization method described in Eqn. (17)-Eqn. (19) yields better numerical results with respect to the ones obtained by the methods in [12], and [16]. It also appears that the rate of conservativeness of the numerical results can be reduced by adopting larger nonuniform delay partitioning limit, N.

Table 1. The maximum allowable upper bound for h_2 versus h_1.

h_1	Methods	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.9$
2.0	Shao (2009)	2.6972	2.5048	2.5048
	Sun *et al.* (2009)	3.0129	2.5663	2.5663
	Theorem 1, $N = 2$	3.1767	2.5339	2.5339
	Theorem 1, $N = 5$	3.1938	2.5412	2.5412
3.0	Shao (2009)	3.2591	3.2591	3.2591
	Sun *et al.* (2009)	3.3408	3.3408	3.3408
	Theorem 1, $N = 2$	3.3475	3.3475	3.3475
	Theorem 1, $N = 5$	3.3574	3.3570	3.3570
4.0	Shao (2009)	4.0744	4.0744	4.0744
	Sun *et al.* (2009)	4.1690	4.1690	4.1690
	Theorem 1, $N = 2$	4.2072	4.2072	4.2072
	Theorem 1, $N = 5$	4.2182	4.2182	4.2182
5.0	Shao (2009)	-	-	-
	Sun *et al.* (2009)	5.0275	5.0275	5.0275
	Theorem 1, $N = 2$	5.0936	5.0936	5.0936
	Theorem 1, $N = 5$	5.1060	5.1060	5.1060

Table 2. The maximum allowable upper bound for h_2 versus h_1 with unknown μ.

Methods	h_1	2.0	3.0	4.0	5.0
Shao (2009)	h_2	2.5048	3.2591	4.0744	-
Sun *et al.* (2009)	h_2	2.5663	3.3408	4.1690	5.0275
Corollary 1, $N = 2$	h_2	2.5339	3.3475	4.2072	5.0936
Corollary 1, $N = 5$	h_2	2.5412	3.3570	4.2182	5.1060

6. Conclusions

The stability problem of a class of interval time-varying delay systems is studied by introducing a technique of a nonuniform delay partitioning for the delay interval. A Lyapunov-Krasovskii functional candidate is selected in accordance with the nonuniformly decomposed pattern of the delay interval so that some sufficient stability criteria are obtained. The proposed methods can handle both delay differentiable case and nondifferentiable delay case or unknown delay derivative bound case. Two suboptimal delay fractionation schemes, namely, linearization with cone complementary technique and linearization under additional constraints are introduced. It is theoretically demonstrated that the proposed technique has reduced complexity in comparison to some existing delay fractionation methods from the literature. In order to illustrate the improvements of the proposed approach, a numerical example is presented and numerical results on the maximum allowable delay bound are exhibited along with a comparison with those of some existing methods from the literature.

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