

# An Efficient Algorithm to Determine the Periodic Steady-State Solutions of Nonlinear Systems Using Extended Spectral Analysis

Gülay Tohumoğlu

University of Gaziantep, Electrical and Electronics,  
Engineering Department, 27310 Gaziantep-TURKEY  
e-mail: gulay@gantep.edu.tr

## Abstract

*A new method for the solution of nonlinear periodic systems was developed. It avoids the time domain calculations of the whole network equations. In the proposed method, by forming the augmented network as linear and nonlinear subnetworks these subnetworks are formulated in complex frequency and time domain respectively. Using spectral analysis the steady-state periodic solution of the whole nonlinear network is reached by an iterative approach. The method can be applied efficiently to weak and strong nonlinear circuits equally well. When we divide the network into a large linear subnetwork and a small nonlinear subnetwork, this method can be applied more efficiently. Solutions depending on the chosen total harmonic numbers converge to the exact solutions.*

## 1. Introduction

An important problem in the computer-aided theory of nonlinear systems is the steady-state analysis of nonlinear circuits with periodic responses. There are 2 basic approaches in the literature, i.e., time-domain approaches [1-5] and frequency-domain approaches [6-13]. In principle, the periodic response, if it exists, can always be found by integrating the system differential equations until the transient responses die out, which is known as the 'brute force method' but it is computationally very expensive in slowly settling systems. Hence better methods are needed.

The shooting methods take the approach of solving a 2-point boundary value problem. The algorithms for this purpose are designed to determine a set of initial conditions after 1 full period [1-3]. A common property of these algorithms is that they involve the numerical integration of the system equations over one or several periods, which is rather time-consuming for large-scale networks. In the time domain, another approach is Köksal's method [4], which is based upon the use of the previous results about the linear time-invariant networks containing periodically operated switches [5]; this is made possible by treating the nonlinear system as a periodically time-varying one under the steady-state conditions. It does not require numerical integrations, which are necessary at least over 1 period in the shooting methods, and it does not use optimization routine programmes as in some of the frequency domain methods.

The most familiar frequency domain method is known as the harmonic balance technique (HBT), which has been reported in many previous papers [6, 7]. The HBT is an iterative technique which seeks to

match the frequency components (harmonics) of a set of variables defined for joining two subnetworks (the linear and nonlinear parts of the network). Hicks and Khan [8] and Camacho-Penalosa [9] have investigated various criteria for convergency conditions of HBT. Basically, the serious disadvantage of these is the large number of unknown variables that must be optimized, although it avoids the computationally expensive process of numerical integration of system differential equations.

Another approach was recently developed by van den Eijnde and Schoukens [10] to determine the steady-state output of the periodically excited system containing strong nonlinearities. It solves this problem by increasing the excitation step by step (which is an approach appearing in Köksal's work as well [4]), and constructing the intermediate solution for some level of excitation from the response corresponding to the preceding step. Actually, the method combines a Volterra approach based upon the algorithm developed by Chua and Ng [11] with some concepts adopted in the well-known harmonic balance technique. Attempts have also been made to develop a general approach to handle nonlinear systems. In addition, Ushida, Adachi, and Chua [12] proposed an algorithm based on both time-domain and frequency-domain approaches. It partitions the large-scale circuit into some large linear subnetworks and small nonlinear subnetworks, and then solving the linear part in the time domain and the weakly nonlinear subnetwork is solved by a frequency-domain approach.

This paper offers an alternative efficient algorithm to calculate the steady-state responses of nonlinear circuits by using the frequency domain techniques, namely, the extended spectral analysis method which is originally derived for periodically linear time-varying systems [13-16]. It uses the advantage of the linear part of the system which is represented by hybrid parameters in the spectral frequency domain. Time-domain equations of the nonlinear part are also transformed into the spectral domain. This new iterative algorithm neither uses optimization nor numerical integration and hence it has advantages over the brute force methods and it forms an alternative for the frequency domain methods.

## 2. Review of Spectral Analysis Method

In a periodically time-varying linear and asymptotically stable system, any steady-state response to the exciting function  $u(t) = Ue^{pt}$ ,  $p = j\omega$  has the form

$$y(t) = Y(p, t)e^{pt} \tag{1}$$

where  $Y(p, t) = H(p, t)U$  and  $H(p, t)$  is the time-varying system function [16] that has the same period as the system ( $T_0$ : system period,  $\omega_0 = 2/T_0$ : system frequency), and  $U$  is a complex value. The time-varying complex amplitude  $Y(p, t)$  can uniquely be represented by a spectral vector  $\mathbf{y}$  containing its Fourier coefficients, i.e.,

$$Y(p, t) = \sum_{n=-\infty}^{\infty} Y_n(p)e^{j\omega_0 nt} \tag{2}$$

and

$$\mathbf{y} = [Y_0 Y_{-1} Y_{+1} Y_{-2} Y_{+2} \dots]^T \tag{3}$$

where supercript  $T$  denotes matrix transpose. This way the response  $y(t)$  is completely represented by the spectral vector  $\mathbf{y}$  of infinite dimension as in the form

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n(p) e^{(j\omega_0 n + p)t} \quad (4)$$

The main idea in the spectral analysis approach is to express any terminal relation  $y(t) = f : x(t)$ ,  $f$  being a linear operator between independent terminal variable  $x(t)$  and the dependent terminal variable  $y(t)$  in the form

$$\mathbf{y} = \mathbf{F}\mathbf{x} \quad (5)$$

where  $\mathbf{y}$  and  $\mathbf{x}$  are the spectral vectors of  $y(t)$  and  $x(t)$  respectively [13-16]. If a signal  $x(t)$  is multiplied by a scalar periodic function  $m(t)$  to form a new signal  $y(t)$ , i.e.,

$$y(t) = m(t)x(t) \quad (6)$$

this relation is expressed in the spectral domain as

$$Y_n = \sum_{l=-\infty}^{\infty} M_{n-l} x_l; \quad n = 0, \pm 1, \pm 2, \dots \quad (7)$$

or equivalently, in matrix form  $\mathbf{y} = \mathbf{M}\mathbf{x}$  where  $\mathbf{M}$  indicates the modem matrix of  $m(t)$ , and  $y$  and  $x$  are spectral vectors of  $y(t)$  and  $x(t)$  respectively. In a similar way, when  $f$  represents the derivative operator, the spectral domain relation can be written as

$$Y_n(p + jn\omega_0) X_n; \quad n = \dots, -2, -1, 0, 1, 2, \dots \quad (8)$$

This also can be written in matrix form as  $\mathbf{y} = \mathbf{D}\mathbf{x}$  where  $\mathbf{D}$  is the diagonal matrix which is called a 'didem matrix'. If  $f$  represents integral operator and the average value of  $x(t)$  is 0, then a similar derivation yields 'indem matrix'  $\mathbf{S}$  which is the inverse of the didem matrix or vice versa.

Starting from the frequency domain relation  $Y(s) = H(s)X(s)$  and transferring it into the spectral domain in the form  $\mathbf{y} = \mathbf{T}\mathbf{x}$  where transfer spectral matrix  $\mathbf{T}$  concerned with  $H(s)$  can be defined as

$$T = \text{diag}[H(p) H(p - j\omega_0) H(p + j\omega_0) H(p - j\omega_0) H(p + j2\omega_0) \dots] \quad (9)$$

It is concluded that all the terminal relations and/or formulations in the time or frequency domain can be simply written in the spectral domain to analyze the whole system [13-17].

### 3. Extension of Spectral Domain Approach to Nonlinear Systems

Consider the relation  $y = f : x(t)$  and its corresponding spectral domain representation  $\mathbf{y} = \mathbf{F}\mathbf{x}$  again. It was previously emphasized that for linear systems the spectral matrix  $\mathbf{F}$  is in general a fixed function of  $j\omega$ . This way, the terminal equations of each component are modelled in the spectral domain, and then the conventional circuit analysis methods (state-space representation, mesh analysis, node analysis etc.) can be used to analyze the whole system [16]. However, for nonlinear components the relation between  $x$  and  $y$  cannot be expressed as in (5) directly with a fixed spectral matrix  $\mathbf{F}$  [13]-[17]; therefore the matrix  $\mathbf{F}$  must be computed for each known independent variable  $x(t)$ .

Concerning the application of spectral analysis to nonlinear networks, it is assumed that the system is excited by periodic forcing functions having a common period  $T_0$ , and the system dynamic is such that

all the steady-state responses are periodic with this common period. This assumption may not be true in some special cases but it is acceptable at present for it makes it possible to treat a nonlinear system as a periodically time-varying linear system [4], [17]. We note that in linear systems the response  $y(t)$  given in (1) is not a periodic function of  $t$ ; however, for nonlinear system due to the above assumption  $Y(p, t)$  and  $e^{pt}$  are both periodic with the common period  $T_0$ ; therefore,  $Y(p, t)$  itself can be used to represent  $y(t)$  or equivalently  $p$  can be treated as 0. With this modification to the spectral theory, it is possible to write all terminal equations defined as  $y(t) = f : x(t)$  where  $f$  is any operator. Thus all derivations given in the previous section are valid.

For steady-state analysis of time-invariant networks it is known that transfer function representation is sufficient. It becomes important to write that relation in terms of spectral vectors of its terminal or port variables. By using previous sections and the above assumption, the transfer spectral matrix  $T$  of transfer function  $H(s)$  can be derived from the frequency domain relation  $Y(s) = H(s)X(s)$  and expressed in the spectral domain as

$$\mathbf{y} = \mathbf{T}\mathbf{x} \text{ with } T = \text{diag} [H(0) H(-j\omega_0) H(j\omega_0) H(-j2\omega_0) H(j2\omega_0) \dots] \quad (10)$$

Transfer function representation is important for us because it will use all the advantages of separating linear and nonlinear parts of the network to apply extended spectral analysis to get the steady-state solution of a nonlinear network.

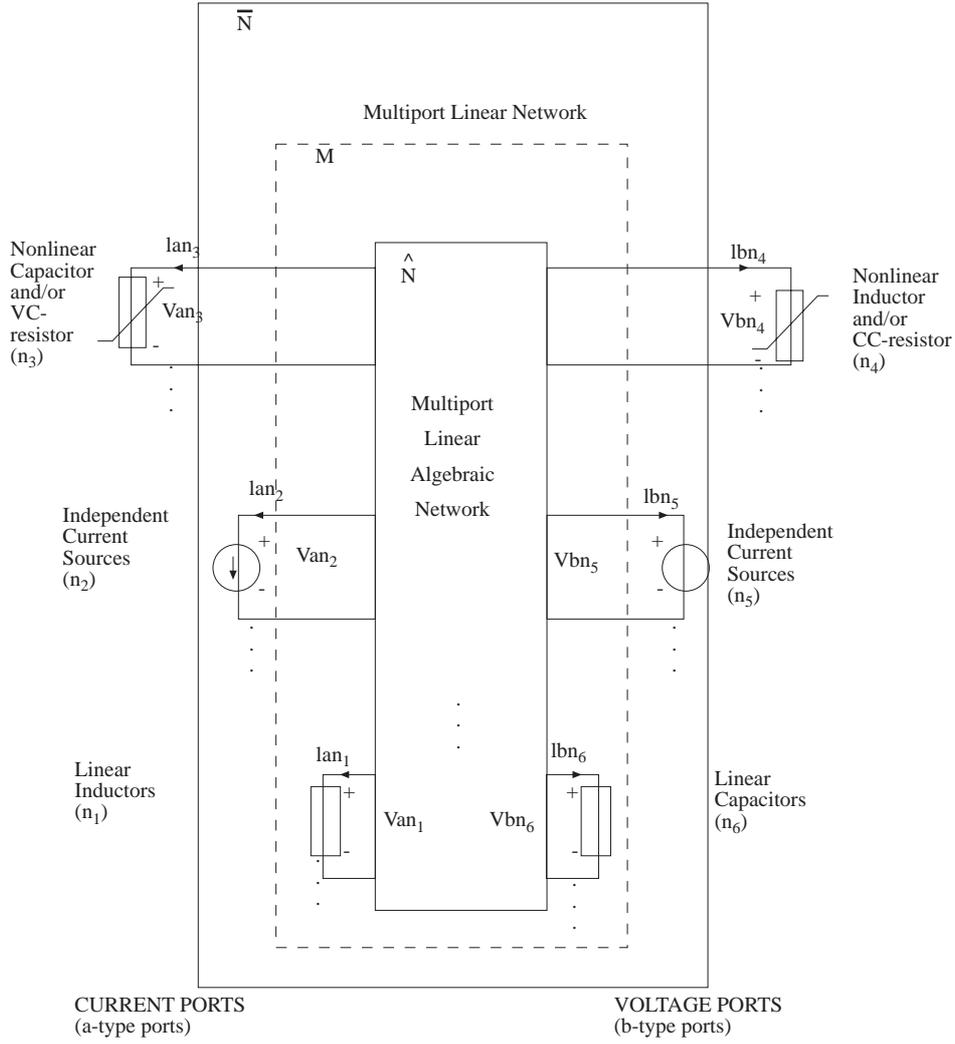
## 4. Representation of Linear and Nonlinear Subnetworks

In many problems in network analysis, the network given contains only a few elements requiring special attention, such as nonlinear elements. The rest of the elements are in general any kind of algebraic components. In such cases it is often advantageous to extract the special elements outside the given network and to regard the remaining linear portion as a linear multi-port. Note that the linear portion is assumed to have no time-varying components and no time delay elements. Further, the algebraic multi-terminal components are assumed to be represented by resistive components and dependent sources.

It has been noted that an extracted element may have more than 2 terminals [18,19]. The extracted elements may be current-controlled (CC) or voltage-controlled (VC) nonlinear elements where they are collected at different sides of the network. As the nonlinear elements, CC and VC resistors, VC capacitors and CC inductors are considered. To make the most efficient use of the linearity the nonlinear network is augmented as shown in Figure 1 [19]. The purpose is to find a hybrid representation of the multi-port linear subnetwork  $\overline{N}$  in the form

$$\begin{bmatrix} V_{an} \\ I_{bn} \end{bmatrix} = [\overline{H}(s)] \begin{bmatrix} I_{an} \\ V_{bn} \end{bmatrix} + [\overline{S}(s)] \begin{bmatrix} I_{aI} \\ V_{bI} \end{bmatrix} \quad (11)$$

where  $I_{an}$  and  $V_{bn}$  ( $V_{an}$  and  $I_{bn}$ ) represent the independent (dependent) port variables and  $I_{aI}$  and  $V_{bI}$  represent the independent current and voltage source vectors.  $\overline{H}(s)$  and  $\overline{S}(s)$  are the transfer spectral matrices obtained by conventional methods [19]. It is worth noting that  $V_{an}$  and  $I_{bn}$  are the independent port variables of the nonlinear voltage and current controlled nonlinear subnetworks, respectively.



**Figure 1.** Augmentation details of the nonlinear network  $N$ ;  $n_i$  shows the number of ports of indicated type,  $i = 1, 2, \dots, 6$ .

To derive a mathematical model for the multi-port linear subnetwork  $\overline{N}$ , as in the form of equation (11), in addition to the extracted elements, linear inductors and capacitors and independent sources are also extracted as shown in Figure 1. Then it is left with only the linear algebraic part  $\hat{N}$  of the original network. Note that the port currents have the same directions as the component currents which are defined in the opposite direction to the conventional notation. This is because sign changes are not needed when the component or port current are considered. The next step is to find a hybrid representation in the form of the subnetwork  $\hat{N}$

$$\begin{bmatrix} V_a \\ I_b \end{bmatrix} = \begin{bmatrix} H_a & H_{ab} \\ H_{ba} & H_b \end{bmatrix} \begin{bmatrix} I_a \\ V_b \end{bmatrix} \quad (12)$$

where the ports are classified as CURRENT ports and VOLTAGE ports, as defined in Figure 1.

It is known that the sufficient conditions for multi-port  $\hat{N}$  to possess a hybrid matrix  $H$  [18-20], as in equation (12), are as follows: i) Controlled voltage sources and/or voltage ports, short circuit elements do not

contain any loops, ii) Controlled current sources and/or current ports, open circuit elements do not contain any cutsets [19]. These conditions when including independent voltage and current sources respectively, are also sufficient for the existence and computation of the hybrid matrix  $\overline{H}$  of  $\overline{N}$ .

Next, terminal equations in terms of the port variables of the extracted elements are defined in (modified) matrix-vector form [20].

For future use the explicit form of the hybrid matrix considered in (11) is obtained by doing some manipulations to define relations between independent and dependent port variables as follows:

$$\begin{bmatrix} V_{aN_3} \\ \vdots \\ I_{bN_4} \end{bmatrix} = \begin{bmatrix} \overline{H}_{11} & \overline{H}_{12} & \vdots & \overline{H}_{13} & \overline{H}_{14} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{H}_{21} & \overline{H}_{22} & \vdots & \overline{H}_{23} & \overline{H}_{24} \end{bmatrix} \begin{bmatrix} I_{an_2} \\ I_{an_3} \\ \vdots \\ V_{bn_4} \\ V_{bn_5} \end{bmatrix} \quad (13)$$

This is the hybrid representation of the multi-port network M indicated in Figure 1. To obtain the hybrid representation  $\overline{N}$  of from (13), we have to put back independent voltage and current source ports and restore their original roles as independent sources inside  $\overline{N}$ . The mathematical counterpart of such a process is to consider  $[I_{an_2} V_{bn_5}]^T$  as known quantities and solving for nonlinear dependent variables. The result has the same form as (11), which can be easily seen by comparing the resultant equation.

With the introduced notation, voltage and current-controlled nonlinear networks, namely, resistive and reactive (capacitive and inductive) networks, are characterized by the following equations.

$$\begin{aligned} I_{aN} &= f_1(V_{aN}, \dot{V}_{aN}) \\ V_{bN} &= f_2(I_{bN}, \dot{I}_{bN}) \end{aligned} \quad (14)$$

In the above functions  $f_1$  and  $f_2$  include  $\dot{V}_{aN}$  and  $\dot{I}_{bN}$  if there are nonlinear capacitors (inductors). These time-domain nonlinear equations are also transferred to the spectral domain by using the necessary (didem) matrix and vector relations previously defined.

### 5. New Algorithm

An algorithm which computes the steady-state solution of nonlinear systems by using extended spectral analysis is described. To start the iterative simultaneous solution of the equations in (11) and (14) first, equation (11) is written in the spectral domain as follows:

$$\begin{bmatrix} V_{aN_0} \\ V_{aN_{-1}} \\ V_{aN_{+1}} \\ \vdots \\ \vdots \\ I_{bN_0} \\ I_{bN_{-1}} \\ I_{bN_{+1}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \overline{H}_{11} & \overline{H}_{12} \\ \overline{H}_{21} & \overline{H}_{22} \end{bmatrix} \begin{bmatrix} I_{aN_0} \\ I_{aN_{-1}} \\ I_{aN_{+1}} \\ \vdots \\ \vdots \\ V_{bN_0} \\ V_{bN_{-1}} \\ V_{bN_{+1}} \\ \vdots \end{bmatrix} + \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} \\ \overline{S}_{21} & \overline{S}_{22} \end{bmatrix} \begin{bmatrix} I_{aI_0} \\ I_{aI_{-1}} \\ I_{aI_{+1}} \\ \vdots \\ \vdots \\ V_{bI_0} \\ V_{bI_{-1}} \\ V_{bI_{+1}} \\ \vdots \end{bmatrix} \quad (15)$$

where

$$\overline{H}_{ij} = \text{diag}[\overline{H}_{ij}(0) \overline{H}_{ij}(-j\omega_0) \overline{H}_{ij}(+j\omega_0) \dots] \quad (16a)$$

$$\overline{S}_{ij} = \text{diag}[\overline{S}_{ij}(0) \overline{S}_{ij}(-j\omega_0) \overline{S}_{ij}(+j\omega_0) \dots] \quad (16b)$$

$$(16c)$$

with  $i, j = 1, 2$ . Equation (14) must also be transferred into the spectral domain depending on the specified nonlinear functions describing nonlinear component behaviour.

The new algorithm can be summarized in the following steps.

i) Choose a set of initial values for the unknown spectral vector appearing in the right-hand side of (15); the spectral vector of excitations in this equation is known.

ii) Compute the unknown dependent variables, spectral vectors on the left-hand side of (15) (spectral vectors of independent terminal variables of nonlinear components in (14)).

iii) Originally, the nonlinear terminal equations are written in terms of spectral vectors, spectral matrices as didem, indem, modem matrices etc. corresponding to the nonlinear equations in the time domain (Eq. 14) in the form of

$$[I_{aN}] = F_1(V_{aN}, \mathbf{D}[\mathbf{V}_{aN}]) \quad (17a)$$

$$[V_{bN}] = F_2(I_{bN}, \mathbf{D}[\mathbf{I}_{bN}]) \quad (17b)$$

where  $\mathbf{D}$  is the didem matrix concerned with the derivative (operator) of the dependent nonlinear variables  $V_{aN}$  and  $I_{bN}$ . Having the computed spectral vectors of independent terminal variables of nonlinear networks, evaluate the dependent terminal variables of the nonlinear components.

iv) Compare the computed spectral vectors with the previous ones (either dependent or independent spectral vectors in (15)), if the difference is not tolerably smaller than a prescribed amount. Then go to step ii, otherwise compute the time-domain correspondence of the latest spectral vector and print the results.

**The second algorithm:** Basically, this algorithm is slightly different from the first algorithm. The main difference is changing the third step of the first algorithm. First, the nonlinear terminal equations must be kept in the time domain. By using the spectral vectors computed in step ii, independent variables of the nonlinear terminal equations must be computed. After computing the dependent nonlinear terminal variables from terminal equations, these variables must be expressed in the spectral domain.

For the initial spectral vector choice in the first step, different ways are followed; each nonlinear element can be linearized and the linearized version of (14) in frequency domain is solved simultaneously with (15). Another approach may be the assumption of a 0 spectral vector for each nonlinear element or, having an idea about the spectral content of the terminal variables, the initial spectral vectors can be assigned intuitively [20]. However, the algorithm may not converge due to strong nonlinearities or peculiar behaviour of some special nonlinear networks. In such cases a slight modification is required to prevent large changes in the values of the independent spectral vector  $X$ , the unknown vector in the right-hand side of (15). For this reason, the new spectral vector  $X_n$  computed in step iv of the algorithm is modified as

$$X_m = X_0 + \beta(X_n - X_0) \quad (18)$$

where  $X_0(X_m)$  the old (modified) value of  $X$ ; and  $\beta$  is the modification constant taking either a real constant value between 0 and 1 or variable and complex value. The case of  $\beta=1$  corresponds with no modification. It must be emphasized that complex valued  $\beta$  is more effective than the real one.

For the first iteration, assign-right-hand side unknown vector in (15) 0 by the treatment of voltage ports short circuit and current ports open circuit, i.e., setting  $I_{an} = 0, V_{bn} = 0$ .

To stop the algorithm, the results of successive iterations are compared with a previously defined constant, alpha, by considering the relative error, RE, between the old and new spectral vectors; the relative error is defined as

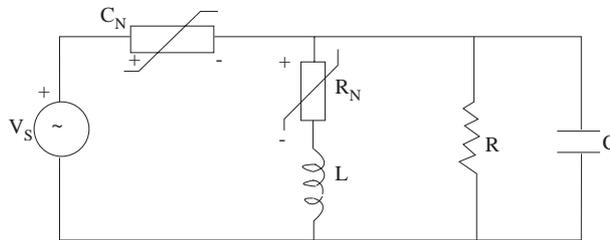
$$RE = \frac{\sqrt{\frac{1}{N} \sum_{i=1}^N (X_{ni} - X_{0i})^2}}{\sqrt{\frac{1}{N} \sum_{i=1}^N X_{ni}^2}} \quad (19)$$

where  $i$  represents the iteration number.

A computer program is prepared to execute the above algorithm. Starting from the topological description of the whole network, this program (SPECNL) contains the evaluation of the transfer characteristics of the linear subnetwork and the application of the algorithm in the preferred version.

## 6. An Example Run

Although there have been many kinds of solved examples, the illustrative example is chosen to neatly present some important features of the developed method. Consider the simple nonlinear network shown in Figure 2.



**Figure 2.** Nonlinear network of Example with the linear component values  $R = 5\Omega, C = 0.1F, L = 0.5H$  and  $V_s(t) = 10 \cos 2\pi t$ .

Note that the nonlinear resistor is current controlled and nonlinear capacitor is voltage controlled as defined by the following component behaviour equations:

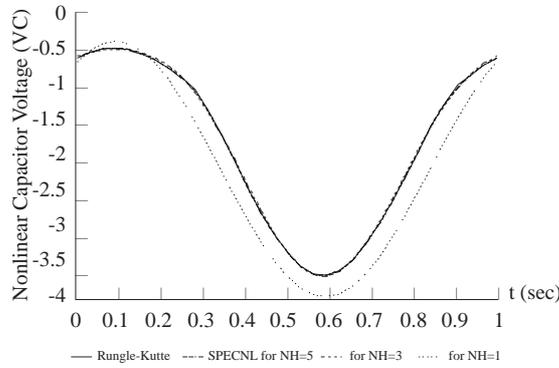
$$\nu_{RN}(i_{RN}) = \begin{cases} 0.5i_{RN} & \text{for } i_{RN} \leq 0 \\ 2i_{RN} - 3i_{RN}^2 + 4i_{RN}^3 & \text{for } i_{RN} > 0 \end{cases} \quad (20)$$

$$Q_C(\nu_{CN}) = \begin{cases} [\nu_{CN}]^{1/2} & \text{for } \nu_{CN} \geq 1 \\ \nu_{CN} & \text{for } -1 < \nu_{CN} < 1 \\ -[-\nu_{CN}]^{1/2} & \text{for } \nu_{CN} \leq -1 \end{cases} \quad (21)$$

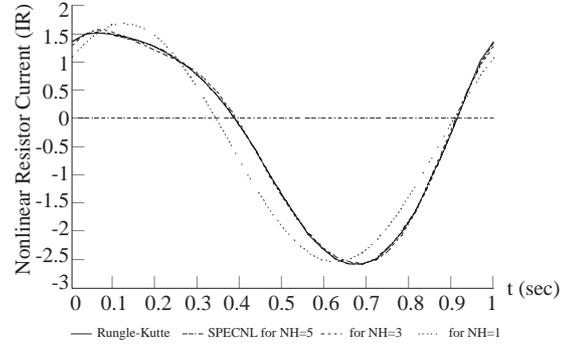
To obtain the actual solution, this example is first analyzed by the brute-force integration method (NSPERK-4th order Runge-Kutta method). Then, applying the extended spectral analysis method and starting from

the 0 initial values for the unknown independent spectral vector in equation (15), the iterations of the algorithm diverge and the actual solution cannot be obtained. However, when the modified version of the algorithm was run to guarantee the convergency,  $\beta=0.1$  was used for the different number of harmonics (NH) 1,3,5,10; the solutions obtained for nonlinear capacitor voltage and resistor current are shown in Figures 3 and 4, respectively. However, instead of using 0 initial values for the unknown spectral vector(s) if it is chosen as  $I_{CN}$  and / or  $V_{NR} = [0 \ 0.5 \ 0.5 \ 0 \ 0 \dots]^T$  (this corresponds  $\cos \omega t$  which is far from the actual solution), the method converges without modification for  $\beta=1$  as well. This implies that, in the case of divergence, instead of using the modified version of the algorithm, several initial spectral vectors may be tried to obtain convergence, which can be expected for nonlinear systems since responses and stability may strongly depend on the initial conditions.

The comparison of the results obtained by the extended spectral analysis method with those from the Runge-Kutta method is also shown in Figures 3 and 4.



**Figure 3.** Nonlinear capacitor voltage versus time.



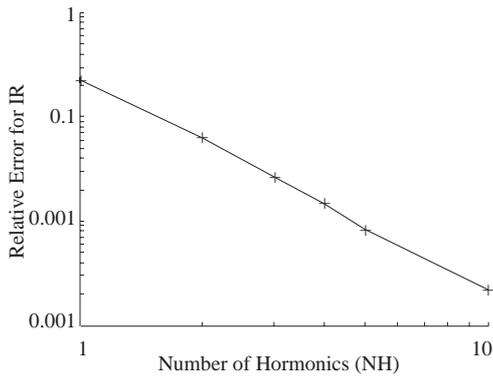
**Figure 4.** Nonlinear resistor current versus time.

For  $NH = 10$ , the numerical results show that SPECNL gives exactly the same values as NSPERK up to 4 decimal places. In this example there are differences between the cases for various  $NH = 1, 3, 5, 10$  values due to strong nonlinear discontinuity that harmonic contents of all orders present in the responses. Obviously, for  $NH = 1$ , there are quite large differences between the values computed by SPECNL and the actual values computed by NSPERK. For  $NH = 3$  and 5 the differences are hardly visible, and they decrease as  $NH$  increases. To show the differences between the results of SPECNL and the true values, the relative error in  $i_{RN}$  versus the number of harmonics is logarithmically plotted in Figure 5. The mathematical relations between  $RE$  and  $NH$  can be easily expressed from this plot as

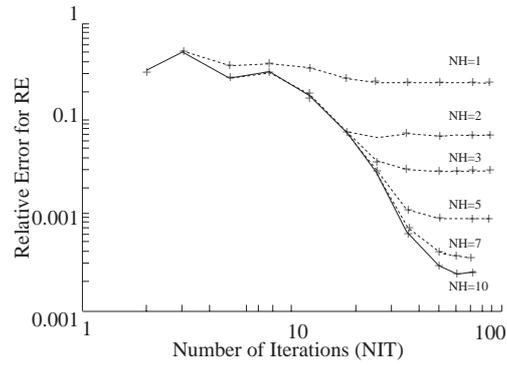
$$\begin{aligned} \ln(RE) &= -1.49 - 2.00 \ln(NH) \text{ or,} \\ RE &= e^{-1.49}(NH)^{-2.00} = 0.225(NH)^{-2.00} \end{aligned} \quad (22)$$

As seen, the relative error is inversely proportional with almost the square of the number of harmonics. As for the computer time, CPU time increases almost linearly with  $NH$  (CPU time  $\approx 2 \cdot NH / 3$  sec.).

In Figure 6 the relative errors with respect to the actual values versus the number of iterations for  $NH = 1, 2, 3, 5, 7, 10$  are plotted in logarithmic scales. Obviously, for each NH the error drops to almost its minimum value so that after a certain number of iterations, error can not be reduced at all.



**Figure 5.** The relative error in  $i_{RN}$  versus number of harmonics in logarithmic scale.



**Figure 6.** Relative error with respect to actual values versus number of iterations for  $NH = 1, 2, 3, 5, 7, 10$ .

When the saturation error against the number of harmonics is drawn, the variation implies that the minimum error that can be attained with the number of harmonics is expressed as

$$RE = 0.226(NH)^{-1.9924} \quad (23)$$

This equation is very similar to the equation given by (22), where the actual error is considered until the error between successive iterations are reduced to a constant value  $10^{-5}$ . This result indicates that  $10^{-5}$  is a proper value to stop the iteration for any higher harmonics.

Another fact observed in Figure 6 is that for a given number of harmonics, it is useless to continue the iterations to improve the results after a certain step until the error reduction is sufficient. Normally, as the number of harmonics decreases, the number of iterations for efficient reduction of the error decreases, because after this number the efficient factor in the reduction of the error becomes the number of harmonics instead of the number of iterations.

## 7. Results and Conclusions

In this research, a new method called extended spectral analysis to compute the periodic steady-state solutions of nonlinear systems is described. It is based upon the separation of linear and nonlinear parts of the network as in the harmonic balance method. However, the application of spectral analysis is originally extended in the sense that not only the steady-state responses of periodically time-varying systems, but also the periodic steady-state responses of nonlinear systems can be computed by spectral domain techniques.

Although the proposed algorithm does not require numerical integration of the system equations and the use of optimization techniques, it needs iterations until a reasonable accuracy has been reached.

The proposed method has been successfully applied to the networks which have weak or strong nonlinearity with good convergence characteristics and performance. However, it is known that, naturally, when a periodic steady-state solution does not exist, the algorithm may not converge to a certain solution. Although the steady-state periodic solution is known to exist, there are cases in which the convergence to this solution is possible only if the modified algorithm is applied. In some situations, the proposed algorithm may not lead to a certain solution, it diverges. However, this is not due to the deficiency of the algorithm all the time since, with the chosen initial conditions, the problem solution itself may be unbounded.

In conclusion, a relatively short computation time for smooth nonlinear cases, and direct overview of the spectral behaviour of the nonlinear system responses, straightforward computation of steady-state

periodic response, good convergence and performance are among the main advantages of the proposed method.

As a further study, it will be determined if the network has time-varying and time-delay elements.

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